Федеральное агентство по образованию

Государственное образовательное учреждение
высшего профессионального образования

Владимирский государственный университет

INTEGRAL EQUATION AND CALCULUIS OF VARIATION IN RADIOENGINERING

Manual of lectures
in english language

ИНТЕГРАЛЬНЫЕ УРАВНЕНИЯ И ВАРИАЦИОННОЕ ИСЧИСЛЕНИЕ В РАДИОЭЛЕКТРОНИКЕ

Конспект лекций
на английском языке

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Владимир 2006

УДК 517.972.517.968
ББК 22.161.67.32
П53

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Печатается по решению редакционно-издательского совета
Владимирского государственного университета

П53 **Интегральные** уравнения и вариационное исчисление в радиоэлектронике : конспект лекций на английском языке / Владим. гос. ун-т. ; сост. П. А. Полушин. – Владимир: Изд-во Владим. гос. ун-та, 2006. – 100 с.

Конспект лекций состоит из двух частей. Часть "Интегральные уравнения" разработана для магистров направления 210300. Она используется при обучении в рамках дисциплин "Математический аппарат современной радиотехники" и "История и методология современной радиоэлектроники".

Рассматриваются вопросы, связанные с продолжением изучения интегрального исчисления и смежных задач. Оно используется для дальнейшего углубленного знакомства магистров с современным математическим аппаратом научных исследований. Курс может быть полезен для студентов дневной формы обучения.

Часть "Вариационное исчисление" выступает как продолжение изучения математического анализа и знакомит с разделами математического аппарата, применяемого в разнообразных областях радиоэлектроники.

Курс предназначен для магистров направления 210300, а также может быть полезен для студентов инженерных специальностей.

УДК 517.972.517.968
ББК 22.161.67.32

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2. CALCULUS OF VARIATIONS

Foreword

Various effective mathematical methods are used rather intensively in modern scientific researches. Behavior of complex systems including nonlinear systems may be described in general by means of integro-differentials equations. In spite of wide usage of computers and numerical methods the solution in analytical form gives more opportunities for researches.

The calculus of variations is nearer mathematical method. It gives opportunities to solve many optimization problems. The search of optimums is the final purpose or one of possible intermediate results.

Both integral equations and calculus of variations are universal methods and can be used not only in radio electronics but in other branches of technique. Every scientist must become proficient in these branches.

В современных научных исследованиях и разработках все шире используются различные эффективные математические методы. Поведение сложных систем, включая нелинейные системы, описывается в общем случае с помощью интегро-дифференциальных уравнений. Несмотря на то, что различные частные решения удается получить с помощью численных методов и применения компьютеров, решение в аналитической форме дает гораздо больше возможностей для исследования.

Близким по математическому аппарату является вариационное исчисление. Его методы дают возможности решения многих оптимизационных задач. Поиск и нахождение экстремумов зачастую является аналогом решения задачи или выступает как один из важных промежуточных результатов.

И интегральные уравнения, и вариационное исчисление - достаточно универсальные методы и могу использоваться не только в радиоэлектронике, но и в других областях техники. Умение их использовать должно входить в научный инструментарий исследователей и научных работников.

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2. CALCULUS OF VARIATIONS

1. INTEGRAL EQUATIONS

Integral equation is a equation, where a unknown function is used as argument of integral.

1.1. Linear integral equations

Suppose:

$f(t)$ is a known function,

$\varphi(t)$ is a unknown function we want to find.

If the unknown function in the integral is linear we say that the integral equation is linear.

Classic view of the linear integral equation is:

$$\varphi(t) = \lambda \int_a^b k(t,S)\varphi(S)dS + f(t),$$

where λ is a parameter defining a family of solutions of the integral equation; $k(t,S)$ is a integral equation kernel.

Function $f(t)$ exists in range $a \le t \le b$.

Function $k(t,S)$ exists in range:

$$\begin{cases} a \le t \le b; \\ a \le S \le b. \end{cases}$$

1.2. Kinds of a linear integral equation

1.2.1. Fredholm's equation

General view of 1st kind of Fredholm's integral equation is:

$$\int_a^b k(t,S)\varphi(S)dS = f(t) \ .$$

General view of 2nd kind of Fredholm's integral equation is:

$$\varphi(t) = \lambda \int_a^b k(t,S)\varphi(S)dS + f(t),$$

a and *b* may be finite or infinite.

Solution of the integral equation exists if following conditions are satisfied:

1) *f(t)* is continuous in range *a, b* and

$$\int_a^b |f(t)|^2 dt < +\infty,$$

2) $k(t, S)$ is continuous in ranges

$$\begin{cases} a \le t \le b; \\ a \le S \le b; \end{cases}$$

and

$$\int_a^b \int |k(t, S)|^2 < +\infty.$$

If equation kernels satisfy above conditions, then we say that those kernels are *Fredholm's*. If $f(t) \equiv 0$, everywhere in range (a, b), then the integral equation is said to be *homogeneous*:

$$0 = \varphi(t) + \lambda \int_a^b k(t, S)\varphi(S)dS.$$

In other case, the equation is *heterogeneous*.

1.2.2. Volterra's equations

Common view of 1st kind of Fredholm's equation:

$$\int_a^t k(t, S)\varphi(S)dS = f(t).$$

Common view of 2nd kind of Fredholm's equation:

$$\varphi(t) + \lambda \int_a^t k(t, S)\varphi(S)dS = f(t).$$

If $f(t) \equiv 0$ then the equation is said to be homogeneous. Under some restrictions, Volterr's equations can be considered as Fredholm's equations. If we change the kernel following way

$$H(t, S) = \begin{Bmatrix} k(t, S), & S \le t \\ 0 \quad , & S > t \end{Bmatrix},$$

then we may get Fredholm's equation:

$$\varphi(t) = \lambda \int_a^b H(t, S)\varphi(S)dS + f(t).$$

1.3. Kinds of a non-linear integral equations

1.3.1. Urysohn's integral equation

$$\varphi(t)=\int_a^b k[t,S,\varphi(S)]dS.$$

The kernel includes the unknown function, assume the function *K(x,y,z)* is continuous for all its arguments.

1.3.2. Gammershtein's equation

$$\varphi(t)=\int_a^b k(t,S)F[S,\varphi(S)]dS,$$

where $k(t,S)$ is Fredholm's kernel.

1.3.3. Liapunov-Likhtenshtein's equation

These integrals include essentially non-linear functions. For example:

$$\varphi(t)=f(t)+\lambda\int_a^b k_1(t,S)\varphi(S)dS+\mu\int_a^b\int_a^b k_2(t,S,z)\varphi(S)\varphi(z)dSdz.$$

The equation may include members with even greater non-linear.

1.3.4. Volterra's non-linear integral equation

$$\varphi(t)=\int_a^t k[t,S,\varphi(S)]dS.$$

K(x,y,z) is continuous for all its arguments.

Examples.

1. $g(x)=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{jxy}f(y)dy.$

This is 1st kind of Fredholm's integral equation with following kernel:

$$k(x,y)=\frac{e^{ixy}}{\sqrt{2\pi}}.$$

View of its solution (got by Fourier in 1811) is:

$$\varphi(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ixy}g(x)dx.$$

2. Solution of a common integral equation leads to Volterra's non-linear integral equation, for example (Cauchy problem):

$$\frac{dx(t)}{dt} = F[t, x(t)].$$

for boundary condition $x(a) = x_0$.

Integrate both part of that expression for *t*, and we get:

$$x(t) = x_0 + \int_a^t F[t, x(t)]dt.$$

This expression is 2nd kind of Fredholm's integral equation.

3. General solution of linear n-th order linear differantial equation.

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_n(t)x(t) = F(t).$$

Initial conditions are:

$$x(a) = c_0;$$

$$x'(a) = c_1;$$

$$\ldots$$

$$x^{(n-1)}(a) = c_{n-1}.$$

This task comes to linear integral equation; suppose for n=2:

$$\frac{d^2 x}{dt^2} + a_1(t)\frac{dx}{dt} + a_2(t)x(t) = F(t).$$

Assume: $$\frac{d^2 x}{dt^2} = \varphi(t), \quad x(0)=C_0, x'(0)=C_1;$$

$$\frac{dx}{dt} = \int_0^t \varphi(S)dS + C_1.$$

We know that:

$$\underbrace{\int_0^t dt \int_0^t dt \ldots \int_0^t f(t)dt}_{n \text{ times}} = \frac{1}{(n-1)!}\int_0^t (t-S)^{n-1} f(S)dS.$$

Hence: $$x(t) = \int_0^t (t-S)\varphi(S)dS + C_1 t + C_0.$$

After we substituted that expression in the original differential equation:

$$\varphi(t) + \int_0^t [a_1(t) + a_2(t)(t-S)]\varphi(S)dS = F(t) - C_1 a_1(t) - C_1 t a_2(t) - C_0 a_2(t).$$

After defining:

$$k(t,S) = -\lfloor a_1(t) + a_2(t)(t-S) \rfloor;$$
$$f(t) = F(t) - C_1 a_1(t) - C_1 t a_2(t) - C_0 a_2(t),$$

we get 2nd kind of Volterra's integral equation:

$$\varphi(t) = \int_0^t k(t,S)\varphi(S) + f(t).$$

Solution of that gives to us the unknown function.

In many cases, kernel $k(t,S)=k(t-S)$ is proportional to the different of the arguments, then Volterra's equation is named *integral equation like convolution*, *Abel's integral equation*:

$$f(x) = \int_0^x \frac{\varphi(S)}{\sqrt{x-S}} dS .$$

If the unknown function is contained both under a sign of derivative and under sign of integral, then this equation is said to be integro-differentual equation (IDE).

1.4. Fredholm's methods

In the beginning of the 20-th century, Fredholm completely investigated integro-differential equations.

Solution of the equation:

$$\varphi(t) = \lambda \int_a^b k(t,S)\varphi(S) + f(t),$$

is considered as analog of the solution of n-order system of linear equations to contain *n* unknown variables. In result, the solution comes approximate and depends on *n.* The more *n*, the more precisely solution.

Solution consists of several stages.

Replace the integral with finite sum.

Divide the whole range [*a,b*] to *n* equal parts. Length of those parts is:

$$\delta = \frac{b-a}{n}.$$

Into each part j we choose some point S_j. We get a set of functions $\varphi(S_j)=\varphi_j$. We aren't looking for continuous function, we are finding a set of discrete values φ_j.

$$\varphi(t) \cong \lambda \sum_{j=1}^{n} k\left(t, S_j\right)\varphi_j \delta + f(t),$$

is in the same range that S, because we can select $t_i=S_i$

$$\varphi\left(S_j\right) = \lambda \sum_{j=1}^{n} k\left(S_i, S_j\right)\varphi_j \delta + f\left(S_j\right).$$

Define:

$$f\left(S_i\right) = f_i \text{ , } k\left(S_i, S_j\right) = k_{ij} \text{ ;}$$

$$\left.\begin{matrix} \varphi_i \cong \lambda\delta \sum_{j=1}^{n} k_{ij}\varphi_j + f_i \\ \vdots \end{matrix}\right\} \; i = 1 \div n,$$

where n is the amount of linear algebraic equations.

Rewrite in more usual representation:

$$\varphi_i - \lambda\delta \sum_{j=1}^{n} k_{ij}\varphi_j = f_i \text{ .}$$

Determinant of this system is:

$$D_n(\lambda) = \begin{vmatrix} 1-\lambda\delta k_{11} & -\lambda\delta k_{12} & \cdots & -\lambda\delta k_{1n}; \\ -k_{21} & 1-\lambda\delta k_{22} & \cdots & -\lambda\delta k_{2n}; \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda\delta k_{n1} & -\lambda\delta k_{n2} & \cdots & 1-\lambda\delta k_{nn}. \end{vmatrix} \text{ - polynomial relative to } \lambda.$$

If $D_n(\lambda) \neq 0$, the system has solution for any f_i and has solution integral equation for any $f(t)$. Solving it, we get a set of $\varphi_j=\varphi_j(S_j)$, which is piecewise-linear approximation the unknown function $\varphi_j(S_j)$. If $D_n(\lambda)=0$, this case is a special.

Fredholm's resolvent

We have a set of $\varphi(S_j)$. Substituting this set in the original equation we get:

$$\varphi(t) \cong \lambda \sum_{j=1}^{n} k\left(t, S_j\right)\varphi\left(S_j\right)\delta + f(t).$$

We can represent the acquired solution of the equation in the following general view:

$$\varphi(t) \cong f(t) + \lambda \frac{Q(t, S_1, S_2 \ldots S_n, \lambda)}{D_n(\lambda)},$$

where Q is result of one way of computing of the solution of the system.

When $n \to \infty$, and kernel $k(t,S)$ is continuous and absolute term is $f(t)$: $Q(t, S_1 \ldots S_n, \lambda) \to \int_a^b D(t, S, \lambda) f(S) dS$.

$$D_n(\lambda) \to D(\lambda).$$

Expression of Fredholm's resolvent is function:

$$R(t, S, \lambda) = \frac{D(t, S, \lambda)}{D(\lambda)}.$$

Using this resolvent we get final solution in compact form:

$$\varphi(t) = f(t) + \lambda \int_a^b R(t, S, \lambda) f(S) dS.$$

Resolvent doesn't depend on absolute term, but resolvent is defining only the kernel. Resolvent is used in cases when we want to investigate a response of the same object to many variants of different forces (*f(S)*).

D(λ) and *D(t,s,λ)* are found by constructing their for different degrees λ and applying limit conversion for $n \to \infty$.

Defining $K(S_i, S_i)=K_{ij}$, we have:

$$D_n(\lambda) = \begin{vmatrix} 1-\lambda\delta K_{11} & \ldots & -\lambda\delta K_{1n} \\ -\lambda\delta K_{21} & \ldots & -\lambda\delta K_{2n} \\ \text{------} & \text{------} & \text{------} \\ -\lambda\delta K_{n1} & \ldots & 1-\lambda\delta K_{nn} \end{vmatrix} =$$

$$= (-\lambda\delta)^n \begin{vmatrix} K_{11}+\varepsilon & K_{12} & \ldots & K_{1n} \\ K_{21} & K_{22}+\varepsilon & \ldots & K_{2n} \\ \text{------} & \text{------} & \text{------} & \text{------} \\ K_{n1} & K_{n2} & \ldots & K_{nn}+\varepsilon \end{vmatrix} = (-\lambda\delta)^n F(\varepsilon),$$

where $\varepsilon = -\dfrac{1}{\lambda\delta}$.

$F(\varepsilon)$ is determinant of the matrix and is power function to ε with the top power is n. Consequently it can be expanded into Taylor series:

$$F(\varepsilon) = F(0) + \frac{F'(0)}{1!}\varepsilon + \frac{F''(0)}{2!}\varepsilon^2 + \ldots + \frac{F^{(n)}(0)}{n!}\varepsilon^n;$$

$$F^{(\ell)}(0) = \frac{dF^{(\ell)}(\varepsilon)}{d\varepsilon^{\ell}}, \text{ for } \varepsilon = 0.$$

Differenting the determinant, it comes to the sum of determinants, but its order is decreasing by one.

$$\frac{\partial}{\partial\varepsilon}\begin{vmatrix} K_{11}+\varepsilon & K_{12} & K_{1n} \\ K_{21} & K_{22}+\varepsilon & K_{2n} \\ K_{31} & K_{32} & K_{33}+\varepsilon \end{vmatrix} = \begin{vmatrix} 1 & K_{12} & K_{13} \\ 0 & K_{22}+\varepsilon & K_{23} \\ 0 & K_{32} & K_{33}+\varepsilon \end{vmatrix} +$$

$$+\begin{vmatrix} K_{11}+\varepsilon & 0 & K_{1n} \\ K_{21} & 1 & K_{2n} \\ K_{31} & 0 & K_{33}+\varepsilon \end{vmatrix} + \begin{vmatrix} K_{11}+\varepsilon & K_{12} & 0 \\ K_{21} & K_{22}+\varepsilon & 0 \\ K_{31} & K_{32} & 1 \end{vmatrix} =$$

$$= \begin{vmatrix} K_{11}+\varepsilon & K_{23} \\ K_{11} & K_{33}+\varepsilon \end{vmatrix} + \begin{vmatrix} K_{11}+\varepsilon & K_{13} \\ K_{31} & K_{33}+\varepsilon \end{vmatrix} + \begin{vmatrix} K_{11}+\varepsilon & K_{12} \\ K_{21} & K_{22}+\varepsilon \end{vmatrix} =$$

(more usually is):

$$= \frac{1}{2}\sum_{\alpha 1=1}^{3}\sum_{\alpha 2=1}^{3}\begin{vmatrix} K_{\alpha 1\alpha 2}+\varepsilon & K_{\alpha 1\alpha 2} \\ K_{\alpha 1\alpha 2} & K_{\alpha 1\alpha 2} \end{vmatrix}.$$

(*ε is equal to zero*)

$$D_n(\lambda) = 1 + \sum_{m=1}^{n}\frac{(-\lambda\delta)^m}{m!} *$$

$$*[\sum_{\alpha 1}^{n}\cdots\sum_{\alpha m=1}^{n}\begin{vmatrix} K_{\alpha 1\alpha 1}+\varepsilon & K_{\alpha 1\alpha 2} & \cdots & K_{\alpha 2\alpha m} \\ K_{\alpha 2\alpha 1} & K_{\alpha 2}K_{\alpha 2}+\varepsilon & \cdots & K_{\alpha 2\alpha m} \\ - - - & - - - & - - - & - - - \\ K_{\alpha m\alpha 1} & K_{\alpha 1\alpha 2} & \cdots & K_{\alpha m\alpha m}+\varepsilon \end{vmatrix}];$$

$$\sum_{\alpha 1=1}^{n} K_{\alpha 1\alpha 2}\delta=\sum_{j=1}^{n} K(S_j S_j)\delta \xrightarrow[n\to\infty]{} \int_a^b K(S,S)dS,$$

where $K(t,S)$ is a trace of the kernel.

Fredholm showed that for $n\to\infty$.

$$\sum_{\alpha 1=1}^{n}\cdots\sum_{\alpha m=1}^{n}\begin{vmatrix} K_{\alpha 1\alpha 1} & K_{\alpha 1\alpha 2} & \cdots & K_{\alpha 2\alpha m} \\ K_{\alpha 2\alpha 1} & K_{\alpha 2}K_{\alpha 2} & \cdots & K_{\alpha 2\alpha m} \\ - - - & - - - & - - - & - - - \\ K_{\alpha m\alpha 1} & K_{\alpha 1\alpha 2} & \cdots & K_{\alpha m\alpha m} \end{vmatrix}\delta^m \to C_m =$$

$$=\int_a^b\int_a^b \begin{vmatrix} K_{\alpha 1\alpha 1} & K_{\alpha 1\alpha 2} & \cdots & K_{\alpha 2\alpha m} \\ K_{\alpha 2\alpha 1} & K_{\alpha 2}K_{\alpha 2} & \cdots & K_{\alpha 2\alpha m} \\ - - - & - - - & - - - & - - - \\ K_{\alpha m\alpha 1} & K_{\alpha 1\alpha 2} & \cdots & K_{\alpha m\alpha m} \end{vmatrix} d\alpha_1 \ldots d\alpha_m,$$

where $D(\lambda)=\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}C_m\lambda^m$ is Fredholm's determinant.

Repeating the same reasoning, we can get following expressions:

$$D(t,S,\lambda)=\sum(-1)^m\frac{\lambda^m}{m!}B_m(t,S);$$

$$B_m(t,S)=\int_a^b\ldots\int_a^b \begin{vmatrix} K(t,S) & K(t,\alpha_2) & \cdots & K(t,\alpha_m) \\ K(\alpha_1,S) & K(\alpha_1,\alpha_1) & \cdots & K(\alpha_1,\alpha_m) \\ - - - & - - - & - - - & - - - \\ K(\alpha_m,S) & K(\alpha_m,\alpha_1 & \cdots & K(\alpha_m\alpha_m) \end{vmatrix} d\alpha_1 \ldots d\alpha_m;$$

Note that:

$$C_n = \int_a^b B_{n-1}(t,t)dt;$$

$$B_0 = K(t,S),$$

where $D(t,S,\lambda)$ is a minor of Fredholm's determinant.

$R(t,S,\lambda) = \dfrac{D(t,S,\lambda)}{D(\lambda)}$ is the resolvent which doesn't depend on the absolute term, and is defined with the kernel of the equation.

All cases are considered for $D(\lambda) \neq 0$. Values of λ satisfy condition $D(\lambda) \neq 0$ to be named *a regular*. Values of λ satisfy condition $D(\lambda) = 0$ to be named *a characteristic*.

There is a homogeneous integral equation:

$\varphi(t) = \lambda \int_a^b K(t,S)\varphi(S)dS + f(t)$ comes from heterogeneous:

$\varphi(t) = f(t) + \lambda \int_a^b R(t,S,\lambda) f(S)dS$ for *f(t)=0.*

Consider two cases:

1. λ is a regular $\rightarrow D(\lambda) \neq 0$, then $\varphi(t) \equiv 0$.
2. λ is a characteristic $\rightarrow D(\lambda) = 0$. In this case we can get the solution $t \neq 0$.

Example.

The kernel of the integral equation is given:

$$K(t,S) = e^{t-s}.$$

Construct its resolvent satisfying following conditions:

$$0 \le t \le 1;\ 0 \le S \le 1;\ a = 0;\ b = 1.$$

Solve C_i and B_i:

$$C_0 = 1, \quad B_0(t,S) = e^{t-S};$$

$$C_1 = \int_0^1 B_0(\alpha_1, \alpha_1)d\alpha_1 = 1;$$

$$B_1(t,S) = \int_0^1 \begin{vmatrix} e^t e^S & e^t e^{-\alpha 1} \\ e^{\alpha 1} e^{-S} & e^{\alpha 1} e^{-\alpha 1} \end{vmatrix} d\alpha_1 = e^t \int_0^1 e^{\alpha 1} \begin{vmatrix} e^{-S} & e^{-\alpha 1} \\ e^{-S} & e^{-\alpha 1} \end{vmatrix} d\alpha_1 = 0;$$

$$C_2 = \int_0^1 B_1(\alpha_1, \alpha_1) d\alpha_1 = 0.$$

Other $B_k, C_k = 0$.

Therefore: $D(\lambda) = 1 - \lambda$;

$$D(t, S, \lambda) = e^{t-S};$$

$$R(t, S, \lambda) = \frac{e^{t-S}}{1-\lambda}.$$

In case $\lambda \neq 1$, solution is: $\varphi(t) = f(t) + \dfrac{\lambda}{1-\lambda} \int_0^1 e^{t-S} f(S) dS$.

1.5. Integral equation with singular kernel

If the kernel of the equation is a singular, then the solution of the equation is even easily.

Expression of the singular kernels is:

$$k(t, S) = \sum_{j=1}^{n} a_j(t) b_j(S).$$

Supposing, *a* and *b* is linear indepented functions. In this case, 2nd kind of Fredholm's integreal equation can be presented as:

$$\varphi(t) = \lambda \int_a^b \sum_{j=1}^{n} a_j(t) b_j(S) \varphi(S) dS + f(t) = \lambda \sum_{j=1}^{n} a_j(t) \int_a^b b_j(S) \varphi(S) + f(t) =$$

$$= \lambda \sum_{j=1}^{n} C_j a_j(t) + f(t), \qquad (1.5.1.)$$

where $C_j = \int_a^b b_j(S) \varphi(S) dS$.

Solution of the integral equation comes to determinating of unknown constants C_j.

Multiply both sides of the equation by b_j and integrate respect to *t*:

$$\underbrace{\int_a^b \varphi(t) b_i(t) dt}_{C_i} = \underbrace{\int_a^b f(t) b_i(t) dt}_{f_i} + \lambda \sum_{j=1}^{n} C_j \underbrace{\int_a^b a_j(t) b_i(t) dt}_{k_{ij}};$$

$$C_i = f_i + \lambda \sum_{j=1}^{n} C_j k_{ij} \;; \Rightarrow \begin{cases} C_1 = f_1 + \lambda \sum_{j=1}^{n} C_j k_{j1}; \\ C_2 = f_2 + \lambda \sum_{j=1}^{n} C_j k_{j2}; \\ \vdots \\ \mathrm{C_n} = f_n + \lambda \sum_{j=1}^{n} C_j k_{jn}. \end{cases}$$

This expression appropriates for all indexes. Solution of the system gives a able to determinate C_i.

$$C_i - \lambda \sum_{j=1}^{n} C_j k_{ij} = f_i \;;\; i = 1 \div n.$$

If the system can not be solved, then original integral equation is unsolved too.

$$D(\lambda) = \begin{vmatrix} 1-\lambda k_{11} & -\lambda k_{12} & \cdots & -\lambda k_{1n}; \\ -k_{21} & 1-\lambda k_{22} & \cdots & -\lambda k_{2n}; \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda k_{n1} & -\lambda k_{n2} & \cdots & 1-\lambda k_{nn}. \end{vmatrix}$$

If $D \neq 0$, then find a solution as usually, i.e. find values of coefficients C_i.

When the coefficients are computed, we substitute their in the equation (5.1.), then we get unknown function $\varphi(t)$.

Example.

$$\varphi(t) = 1 + \lambda \int_0^1 (t - S)\varphi(S)dS\,;$$

$$k(t,S) = t - S;\; a_1(t) = t;\; a_2(t) = 1;\; b_1(S) = 1;\; b_2(S) = -S;$$

$$\varphi(t) = 1 + \lambda t \int_0^1 \varphi(S)dS + \lambda \int_0^1 (-S)\varphi(S)dS\,;$$

$$C_1 = \int_0^1 \varphi(S)dS; \quad C_2 = \int_0^1 (-S)\varphi(S)dS;$$

$$\varphi(t) = 1 + \lambda C_1 t + \lambda C_2.$$

Multiply both sides of this equation by b_1 and by b_2, then integrate respect to *t*:

$$\begin{cases} \int_0^1 \varphi(t)dt = \int_0^1 dt + \lambda C_1 \int_0^1 tdt + \lambda \int_0^1 dt; \\ \int_0^1 (-1)\varphi(t)dt = \int_0^1 (-t)dt + \lambda C_1 \int_0^1 \left(-t^2\right)dt + \lambda C_2 \int_0^1 (-t)dt. \end{cases}$$

$$\begin{cases} C_1\left(1-\frac{\lambda}{2}\right) - \lambda C_2 = 1; \\ C_1 \frac{\lambda}{3} + \left(1+\frac{\lambda}{2}\right)C_2 = -\frac{1}{2}. \end{cases}$$

$$D(\lambda) = \begin{vmatrix} 1-\frac{\lambda}{2}; & -\lambda \\ \frac{\lambda}{3} & ; \ 1+\frac{\lambda}{2} \end{vmatrix} = 1 + \frac{\lambda^2}{12}.$$

This integral equation has solution anyway, because $D \neq 0$ for all real λ.

$$C_1 = \frac{12}{12+\lambda^2}; \quad C_2 = -\frac{6+\lambda}{12+\lambda^2};$$

$$\varphi(t) = \frac{6(2-2\lambda t-\lambda)}{12+\lambda^2}.$$

For these equation resolvent is a rational function.

1.6. The usage of singular kernels for approximate solving integral equations

Assume, we have some integral equation with non degenerate kernel *k(t,S)*.

$$\varphi(t) = \lambda \int_a^b k(t,S)\varphi(S)dS + f(t).$$

In the integrating range, non degenerate kernel is substituted approximate singular kernel. In this case, approximate solution is enough close to truly solution. The more close approximation, the more correct solution.

In the most cases, power polynomial or trigonometrical functions are used as approximation.

Example.

$$\varphi(t)=\int_0^1 t\left(1-e^{tS}\right)\varphi(S)dS+e^t-t.$$

Exact solution is $\varphi(t)\equiv 1$. Get approximate solution $\varphi_{app}(t)$ *for approximate kernel*.

$$k_b(t,S)=-t^2S-\frac{t^3S^2}{2}-\frac{t^4S^3}{6}.$$

Approximate solution:

$$\varphi(t)=e^t-t-0{,}5t^2-0{,}17t^3-0{,}04t^4.$$

In the range [0;1], the error respect to exact solution is just 0,8%.

1.7. Method of sequential approximation («compressed representations»)

We make sequence of functions. First function is any. Further we make next function from previous, etc.

Following conditionals must be satisfied:

1) In the square *a≤t,S≤b* kernel *k(t,S)* must be continuous and constrained.
2) Declare.

$$M_0=\max_{t,S\in a\div b}\left|k(t,S)\right|.$$

Bellow conditional must be satisfied:

$$\lambda<\frac{1}{M_0(b-a)}.$$

If all those conditionals are satisfied, then series of sequential approximations is constructed using following rule:

$$\varphi_{n+1}(t)=\lambda\int_a^b k(t,S)\varphi_n(S)dS+f(t).$$

The more number of iteration *n*, the more accuracy solution.

Example.

Solve the integral equation using considered method.

$$\varphi(t)=\frac{5}{6}t+\frac{1}{2}\int_0^1 tS\varphi(S)dS.$$

Kernel $K(t,S)=tS$ is continuous function:

$$\max_{t,S\in 0,1}\left|K(t,S)\right|=1=M_0 .$$

Check we can apply above method:

$$\frac{1}{M_0(b-a)}-1>\frac{L}{2}=\lambda .$$

Both conditionals are satisfied.

Choose first function: $\varphi_0(t)=0$. Further:

$$\varphi_1(t)=\frac{5}{6}t+\frac{t}{2}\int_0^1 S\varphi_0(S)dS=\frac{5}{6}t ;$$

$$\varphi_2(t)=\frac{5}{6}t+\frac{t}{2}\int_0^1 S\frac{5}{6}SdS=\frac{5}{6}t(1+\frac{1}{6}) ;$$

$$\varphi_n(t)=\frac{5}{6}t(1+\frac{1}{6}+\frac{1}{6^2}+...+\frac{1}{6^{n-1}})=t(1-\frac{1}{6^n}) ;$$

$$\varphi(t)=\lim_{n\to\infty}\varphi_n=t .$$

The speed of convergence greatly depends of start approximation. Good choice of approximation can reduce time of solving.

1.8. Using of sequential approximations method for solving 2nd kind of Volterra's integral equations

$\varphi(t)=\lambda\int_a^t K(t,S)\varphi(S)dS+f(t)$ is 2nd kind of Volterra's integral equation.

That kind of equations may be considered as particular case of Fredholm's integral equation, if $K(t,S)=0$ for $S>t$. Different is comparison with λ to be not necessary (λ is any).

<u>Example.</u>

Find unknown function φ, satisfying equation:

$$\varphi(t)=t-\int_a^t (t-S)\varphi(S)dS .$$

Solution.

Assume: $\varphi_0 = 0$ then $\varphi_1(t) = t$;

$$\varphi_2(t) = t - \int_a^t (t-S)SdS = t - \frac{t^3}{3!};$$

$$\varphi_n(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + ... + (-1)^{n-1}\frac{t^{2n-1}}{(2n-1)!};$$

$$\varphi(t) = \lim_{n\to\infty} \varphi_n = \sin t.$$

<u>Example:</u>

With the help of a examined method to slove an integral equation:

$$\varphi(t) = \frac{5}{6}t + \frac{1}{2}\int_0^1 tS\varphi(S)dS.$$

Kernel $K(t,S) = tS$ - function continuous

$$\max_{t,S\in 0,1}|K(t,S)| = 1 = M_0.$$

We will to correct the using method:

$$\frac{1}{M_0(b-a)} - 1 > \frac{L}{2} = \lambda.$$

Both conditions are satisfied:

The firest function : $\varphi_0(t) = 0$.

$$\varphi_1(t) = \frac{5}{6}t + \frac{t}{2}\int_0^1 S\varphi_0(S)dS = \frac{5}{6}t;$$

$$\varphi_2(t) = \frac{5}{6}t + \frac{t}{2}\int_0^1 S\frac{5}{6}SdS = \frac{5}{6}t(1+\frac{1}{6});$$

$$\varphi_n(t) = \frac{5}{6}t(1+\frac{1}{6}+\frac{1}{6^2}+...+\frac{1}{6^{n-1}}) = t(1-\frac{1}{6^n});$$

$$\varphi(t) = \lim_{n\to\infty} \varphi_n = t.$$

The speed of convergence strongly depends on initial approximation. The successful selection of approximation can reduce time of the solution.

1.9. Application of the method Successive approximationses for the solution of integral equations <u>*Volterra*</u> of 2 kind

$\varphi(t) = \lambda \int_a^t K(t,S)\varphi(S)dS + f(t)$ - integral equations <u>***Volterra***</u> of 2 kind.

The similar equations can be considered as a particular case of Fredholm equations.

If $K(t,S) = 0$ when $S > t$. The difference is, that the matching with λ Is not necessary (λ - anyone).

Example:

Find a unknown function φ, satisfying an equation:

$$\varphi(t) = t - \int_a^t (t-S)\varphi(S)dS.$$

Solution:

Let's assume: $\varphi_0 = 0$ then $\varphi_1(t) = t$;

$$\varphi_2(t) = t - \int_a^t (t-S)SdS = t - \frac{t^3}{3!};$$

$$\varphi_n(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots + (-1)^{n-1}\frac{t^{2n-1}}{(2n-1)!};$$

$$\varphi(t) = \lim_{n\to\infty} \varphi_n = \sin t.$$

Application of a method of the approximated solutions for the solution of some kinds of non-linear integral equations:

We have an equation:

$$\varphi(t) = \lambda \int_a^b K[t,S,\varphi(S)]dS + f(t).$$

Conditions of applicability of a method:

1) $f(t)$ Should be a continuous function, $K(t,S,\varphi(S))$. Should be a continuous function on all three arguments.

2) The kernel should to satisfy the conditions the (Lipschitz):

$$\left|K(t,S,Z_2) - K(t,S,Z_1)\right| \le L\left|Z_2 - Z_1\right|.$$

L - The constant of the (Lipschitz), which satisfying the condition.

$$|\lambda| < \frac{1}{L(b-a)}.$$

L usually take minimumly:

$$L_{\min} = \min \frac{|K(t,S,Z_2 - K(t,S,Z_1)|}{|Z_2 - Z_1|}.$$

Then the solution can also be received by a successive approximations by the formula:

$$\varphi_{n+1}(t) = \lambda \int_a^b K[t,S,\varphi_n(S)]dS + f(t).$$

initial approximation $\varphi_0(t)$ - anyone.

Example:

Solve an integral equation of a kind:

$$\varphi(t) = \frac{1}{3}\int_{-1}^{1} \frac{tS}{1+\varphi^2(S)} + 1;$$

$$|\lambda| < \frac{1}{L(b-a)} = \frac{1}{2}, L_{\min} = 1.$$

The necessary conditions are satisfied:

$$\varphi_0(t) = 1; \ \varphi_1(t) = \frac{1}{3}\int_{-1}^{1} \frac{tS}{1+1} dS + 1 = 1; \ \varphi_2(t) = 1; \ \varphi_3(\mathrm{t}) = 1.$$

Solution: $\varphi_0(t) = 1.$

If kernel *k(t,S,z)*- has a restricted derivative on z.

That L can be selected from a condition:

$$\left|\frac{dk}{dz}\right|_{\substack{a \le t, S \le b, \\ -\infty < z < +\infty}} \le L.$$

1.10. Solution of a system of integral equations

There are cases, when it is required to find some unknowns of functions, which:

1) Are determined by integral relations.
2) Are determined connected among themselves.

Writing unknowns of functions, are named as a system of integral equations:

$$\begin{cases} \varphi_1(t) = \lambda \sum\limits_{j=1}^{n} k_{1j}(t,S)\varphi_j(S)dS + f_1(t); \\ \varphi_2(t) = \lambda \sum\limits_{j=1}^{n} k_{2j}(t,S)\varphi_j(S)dS + f_2(t); \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi_N(t) = \lambda \sum\limits_{j=1}^{n} k_{Nj}(t,S)\varphi_j(S)dS + f_N(t). \end{cases}$$

From an interval $a \le t \le b$ Pass to an interval $a \le t \le a+N(b-a)$.

From a function set $f_1(t) \ldots f_N(t), a \le t \le b$ Pass to "a unified" function $F(t)$:

$$F(t) = f_i[t-(i-1)(b-a)].$$

From a set $\varphi_1 \ldots \varphi_N$ Also pass to <<unified>> $\Phi(t)$:

$$\Phi(t) = \varphi_i[t-(i-1)(b-a)].$$

On an interval:

$$a+(i-1)(b-a) \le t < a+i(b-a).$$

From set k_{ij} pass to:

$$k_C(t,S) = k_{ij}[t-(i-1)(b-a), S-(j-1)(b-a)].$$

when:

$$\begin{cases} a+(i-1)(b-a) \le t < a+i(b-a); \\ a+(j-1)(b-a) \le S < a+j(b-a). \end{cases}$$

Then it is possible to write by one equation all of systems:

$$\Phi(t) = \lambda \int\limits_{a}^{a+N(b-a)} k_C(t,S)\Phi(S)dS + F(t).$$

Further equation is solve by one of known ways.

1.11. Using of the linear operators

Operator: This any operation which transforming elements of one set to units of other set.

$$E_0 \to E_1.$$

The operator A is named linear, if two conditions satisfied:

1) $A(x+y) = A(x) + A(y).$

2) $A(\alpha x) = \alpha A(x);\ \alpha - const.$

$y(t) = \int_a^b k(t,S)x(S)dS$ - the integral operator of the Fredholm above x.

The combination of the linear operators will (derivate-образует) a vector space of the operators.

The operator transforming elements of set in it self ($E \to E$) is named as a unity operator and is meant.

$Ix \to x$ If the return operator exists, the use him to the source operator should give a single operator: $A^{-1}A = I$.

The operator I+A - always has the return operator.

$$S = (I + A)^{-1};$$

$$S(I + A) = I.$$

The decomposition is fair(Under certain conditions).

$$S = (I + A)^{-1} = I - A + A^2 - A^3 + A^4 - A^5 + \dots$$

It will be used for the solution of integral equations.

Let's consider an integral equation of the (Fredholm) of 2 kind:

$$\varphi(t) = \lambda \int_a^b k(t,S)\varphi(S)dS + f(t).$$

Let's designate linear operation:

$$A\varphi = \int_a^b k(t,S)\varphi(S)dS.$$

In the operator form an initial integral equation:

$$\varphi = \lambda A\varphi + f.$$

Sent $A\varphi$ to the left and carrying φ out brackets:

$$(I - \lambda A)\varphi = f.$$

Under certain conditions on the norm of the operator A the solution of an integral equation looks like:

$$\varphi = (I - \lambda A)^{-1} f .$$

The characteristics is known:

$\varphi = f + \lambda A f + \lambda^2 A^2 f + \ldots + \lambda^n A^n f + \ldots$(Neumann) series.

For this decomposition it is necessary:

1) A series should be convergent.
2) That the inequality was executed.

$$\lambda < \frac{1}{\left[\max\limits_{t,S \in a \div b} k(t,S)\right](b-a)} .$$

Let's consider degrees of the operator:

$$A^2 f = A(Af) = \int_a^b k(t,S)\left[\int_a^b k(S,\tau) f(\tau) d\tau\right] dS = \int_a^b\int_a^b k(t,S)k(S,\tau) f(\tau) dS d\tau =$$

$$= \int_a^b \left[\int_a^b k(t,S)k(S,\tau)dS\right] f(\tau) d\tau = \int_a^b k_2(t,S) f(S) dS,$$

when: $k_2(t,\tau) = \int_a^b k(t,S)k(S,\tau)dS$ - repeated kernel similarly.

$$A^3 f = \int_a^b \left[\int_a^b K(t,S)K_2(S,\tau)dS\right] f(\tau)d\tau = \int_a^b K_3,(t,S) f(S)dS;$$

$$K_3(t,S) = \int_a^b K(t,S)K_2(S,\tau)d\tau;$$

$$K_n(t,S) = \int_a^b K(t,S)K_{n-1}(S,\tau)d\tau;$$

$$A^n f = \int_a^b K_n(t,S) f(S)dS.$$

then:

$$\varphi(t) = f(t) + \lambda \int_a^b K_1(t,S) f(S)dS + \lambda^2 \int_a^b K_2(t,S) f(S)dS +$$

$$+ \lambda^3 \int_a^b K_3(t,S) f(S)dS + \ldots = f(t) + \lambda \int_a^b \left[K_1(t,S) + \lambda K_2(t,S) + \lambda^2 K_3(t,S) + \ldots\right] \times$$

$$\times f(s)dS.$$

It can be write, as:

$$\varphi(t) = f(t) + \lambda \int_a^b R(t, S, \lambda) f(S) dS$$ - finding $\varphi(t)$ through a resolving.

The resolving is determined as follows:

$$R(t, S, \lambda) = K_1(t, S) + \lambda K_2(t, S) + \lambda^2 K_3(t, S) + \ldots$$

It satisfying to such properties:

$$R(t, S, \lambda) = K(t, S) + \lambda \int_a^b K(t, \tau) R(\tau, S, \lambda) d\tau;$$

$$R(t, S, \lambda) = K(t, S) + \lambda \int_a^b K(\tau, S) R(t, \tau, \lambda) d\tau.$$

For a resolving following expression also are fair:

$$R(t, S, \lambda_1) - R(t, S, \lambda_2) = (\lambda_1 - \lambda_2) \int_a^b R(t, \tau, \lambda_2) R(\tau, S, \lambda_1) d\tau;$$

$$R(t, S, 0) = K(t, S);$$

$$\frac{\partial R(t, S, \lambda)}{d\lambda} = \int_a^b R(t, \tau, \lambda) R(\tau, S, \lambda) d\tau.$$

The received results are applicable so for equations ***Volterra.***

Example:

1. Solve an integral equation:

$$\varphi(t) = \lambda \int_0^1 tS\varphi(S) dS + f(t);$$

$$K(t, S) = tS, \quad a = 0, \quad b = 1;$$

$$\max |K(t, S)| = 1 \quad \text{при} \quad 0 \le t, S \le 1.$$

The conditions of applicability will be executed.
Let's find sequence of iterated kernels:

$$K_1(t, S) = tS;$$

$$K_2(t, S) = \int_0^1 K(t, \tau) K(\tau, S) d\tau = \int_0^1 t\tau\tau d\tau = \frac{ts}{3};$$

$$K_3(t, S) = \frac{tS}{3^2};$$

$$K_n(t,S)=\frac{tS}{3^{n-1}};$$

$$R(t,S,\lambda)=ts+\frac{\lambda}{3}tS+\frac{\lambda^2}{3^2}+...+\frac{\lambda^n}{3^n}tS+...+=\frac{3tS}{3-\lambda}.$$

Thus, the general solution of an input equation looks like:

$$\varphi(t)=f(t)+\int_0^1\frac{3tS}{3-\lambda}f(S)dS.$$

2. Solve an integral equation:

$$\varphi(t)=e^t+\int_0^t e^{t-S}\varphi(S)dS;$$

$$\lambda=1;$$

$$K_1(t,S)=e^{t-S};$$

$$K_2(t,S)=\int_S^t e^{t-\tau}e^{\tau-S}d\tau=e^{t-S}(t-S);$$

$$K_3(t,S)=e^{t-S}\frac{(t-S)^2}{2!};$$

$$K_n(t,S)=e^{t-S}\frac{(t-S)^{n-1}}{(n-1)!};$$

$$R(t,S,1)=e^{t-S}+...+e^{t-S}\frac{(t-S)^{n-1}}{n!}+...+=e^{2(t-S)}.$$

$\varphi(t)=e^t+\int_0^t e^{2(t-S)}e^S dS=e^{2t}$ - the solution of an integral equation.

1.12. Integral equations with a kernel having a weak feature

The similar equations have a kind of core:

$$K(t,S)=\frac{H(t,S)}{(t-S)^{\alpha}};$$

$$0<\alpha<1.$$

(equation Abel $\int_0^t \frac{\varphi(S)}{\sqrt{t-S}} dS = f(t)$).

Let's consider an appropriate equation ***Volterra***. A general view of an equation:

$$\varphi(t) = f(t) + \int_a^t \frac{H(t,S)}{(t-S)^{\alpha}} \varphi(S) dS,$$

$a \le t \le b$, $S < t$,

when: $\alpha \ge \frac{1}{2}$ Square of a kernel – nonintegrable.

However to solve an equation it is possible. For the solution will use following procedures:

1. Evaluate iterated kernel:

$$K_2(t,S); \quad K_3(t,S), \text{ и т. д.}$$

$$K_2(t,S) = \int_S^t \frac{H(t,\tau)H(\tau,S)}{(t-\tau)^{\alpha}(t-S)^{\alpha}} = (t-S)^{1-2\alpha} F_2(t,S);$$

$$K_3(t,S) = (t-S)^{2-3\alpha} F_3(t,S);$$

$$K_4(t,S) = (t-S)^{3-4\alpha} F_4(t,S).$$

Let's repeat calculation n of time while the nonintegrable component becomes integrated:

$$n(1-\alpha)>1.$$

2. Input equation is lead to an integral equation with iterated kernels. By contraction of both parts with a function $\lambda K(t,S)$.The integral operator of a kind is used for this purpose to both parts of an equation:

$$\lambda \int_a^t K(t,S)(\cdot) dS.$$

Then:

$$\lambda \int_a^t K(t,S)\varphi(S)dS = \lambda \int_a^t K(t,S)f(S)dS + \lambda^2 \int_a^t K(t,S)[\int_a^S K(S,\tau)\varphi(\tau)d\tau] \times$$

$$\times dS = \lambda \int_a^t K(t,S) f(S) dS + \lambda^2 \int_a^t K_2(t,S)\varphi(S) dS. \qquad (1.12.1)$$

But from an input equation:

$$\lambda \int_a^t K(t,S)\varphi(S) dS = \varphi(t) - f(t).$$

Let's put it in (1.12.1):

$$\varphi(t) = \lambda^2 \int_a^t K_2(t,S)\varphi(S) dS + f_2(t);$$

$$f_2(t) = f(t) + \lambda \int_a^t K(t,S) f(S) dS.$$

It is similarly possible to receive:

$$\varphi(t) = \lambda^3 \int_a^t K_3(t,S)\varphi(S) dS + f_3(t),$$

where $$f_3(t) = f_2(t) + \lambda \int_a^t K(t,S) f_2(S) dS,$$

we will proceed until reach n, which have found on 1-м a stage. Thus:

$$\varphi(t) = \lambda^n \int_a^t K_n(t,S)\varphi(S) dS + f_n(t),$$

kernel of this equation K_n - integrated.

The function f_n can be found, therefore obtained equation is decided by usual methods.

1.13. An equation such as a convolution

It is such integral equations, which kerne depends on a difference of arguments. They look like the following:

$$\varphi(t) = \lambda \int_{-\infty}^{\infty} k(t-S)\varphi(S) dS + f(t).$$

1.13.1. Using Fourier transform generally

For the solution of equations such as a convolution the Fourier transform in the following form will be used:

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt;$$

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega.$$

The convolution of functions is following:

$$\Psi(t)=\int_{-\infty}^{\infty} f_1(t)f_2(\tau-t)dt=f_1*f_2.$$

* - The denotation of a convolution.

The integral operator of the Fourier we shall signify *F*(*).

$$F(\omega)=F[f(t)]=Ff.$$

The Fourier transform from a convolution of functions is equal (In view of a constant factor) To product of separate Fourier transforms from each function:

$$F[f_1*f_2]=\sqrt{2\pi}F[f_1]\cdot F[f_2].$$

Let's consider an equation:

$$\varphi(t)=\lambda\int k(t-S)\varphi(S)dS+f(t).$$

Also is applicable to it a Fourier transform:

Designate: $F[\varphi]=\Phi;\ F[f]=F;\ \ F[k]=K.$

Then after a Fourier transform:

$$\Phi(\omega)=\lambda\sqrt{2\pi}K(\omega)\Phi(\omega)+F(\omega),$$

now it is possible to find $\Phi(\omega)$:

$$\Phi(\omega)=\frac{F(\omega)}{1-\lambda\sqrt{2\pi}K(\omega)}.$$

Having taken reconversion of the Fourier, we receive a required function:

$$\varphi(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{F(\omega)e^{j\omega t}}{1-\lambda\sqrt{2\pi}K(\omega)}d\omega.$$

It is possible to use other path.

Let $R(t,\lambda)$-This reconversion of the Fourier from a following function:

$$\frac{K(\omega)}{1-\lambda\sqrt{2\pi}K(\omega)};$$

$$R(t,\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{K(\omega)}{1-\lambda\sqrt{2\pi}K(\omega)}e^{j\omega t}d\omega.$$

Then the solution can be found by the formula:

$$\varphi(t)=f(t)+\lambda\int_{-\infty}^{\infty}R(t-S,\lambda)f(S)dS.$$

1.13.2. Application of a method convolution for the solution of integral equations of 1-st kind

Let it is necessary to solve an equation:

$$\int_{-\infty}^{\infty}k(t-S)\varphi(S)dS=f(t).$$

Is applicable a Fourier transform to both parts and we use properties of a convolution.

After transformation:

$$\sqrt{2\pi}K(\omega)\Phi(\omega)=F(\omega).$$

We have found that: $\varphi(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\omega)}{K(\omega)}e^{j\omega t}d\omega.$

The Laplace transformation is possible also to apply as well as Fourier transform, but it is necessary always at the solution to check up a range of definition.

Example:

$$\varphi(t)=t+\int_{0}^{t}\sin(t-S)\varphi(S)dS,$$

L{}*- The Laplace transformation.

It is known:

$$L\{t\}=\frac{1}{p^2};\quad L\{\sin t\}=\frac{1}{p^2+1};\quad L\{\varphi\}=\Phi(p).$$

Having applied to an equation a Laplace transformation, we shall receive:

$$\Phi(p)=\frac{1}{p^2}+\frac{\Phi(p)}{p^2+1}; \;\Rightarrow\; \Phi(p)=\frac{1}{p^2}+\frac{1}{p^4}.$$

Solution:

$$\varphi(t)=t+\frac{t^3}{3!}.$$

1.13.3. Solution of a system of integral equations

Let we have a system N of integral equations ***Volterra*** of a following kind:

$$\varphi(t)=f_i(t)+\lambda\sum_{j=1}^{n}\int_0^t k_{ij}(t-S)\varphi_j(S)dS; \quad i=1\div n.$$

Is applicable to all equations of this system a Laplace transformation:

$$\Phi_i(p)=F_i(p)+\lambda\sum_{j=1}^{n}k_{ij}(p)\Phi_j(p).$$

Solution this system of algebraic equations as a set of the imagery and finding from them the originals, we shall receive the solution:

$$\Phi_i(p)\Rightarrow\varphi_i(t).$$

1.13.4. Solution of non-linear integral equations

The method is applicable and for some non-linear integral equations. For example:

$$\varphi(t)=\lambda\int_0^t\varphi(S)\varphi(t-S)dS+f(t).$$

This non-linear equation such as a convolution. Is applicable a Laplace transformation to both parts of this equation:

$$\Phi(p)=\lambda\Phi^2(p)+F(p).$$

This quadratic equation, its solution:

$$\Phi(p)=\frac{-1\pm\sqrt{1-4\lambda F(p)}}{2\lambda}.$$

Having taken reconversion of the Laplace, we shall receive $\varphi(t)$.

1.13.5. Solution of integro-differential equations such as a convolution

Following integro-differential equation let is given:

$$\frac{d^n\varphi(t)}{dt^n}+a_1\frac{d^{n-1}\varphi(t)}{dt^{n-1}}+\cdots+a_n\varphi(t)+\sum_{m=0}^{l}\int_0^t k_m(t-S)\left[\frac{d^m\varphi(S)}{dS^m}\right]dS=f(t).$$

Let's designate a set of the initial conditions:

$$\varphi(0)=\varphi_0;\quad \varphi'(0)=\varphi_0';\quad \ldots\quad \varphi^{(n-1)}(0)=\varphi_0^{(n-1)}.$$

Using the following property of a Laplace transformation (for an arbitrary function φ):

$$\frac{d^k\varphi}{dt^k}\Rightarrow p^k\Phi(p)-p^{k-1}\varphi_0-p^{k-2}\varphi_0'-p^{k-3}\varphi_0''-\ldots-\varphi_0^{(k-1)}.$$

Is applicable this property to our equation:

$$\int_0^t k_m(t-S)\varphi^m(S)dS\Rightarrow k_m(p)\left[p^m\Phi(p)-p^{m-1}\varphi_0-\ldots-\varphi_0^{(m-1)}\right].$$

Now equation looks like the following:

$$\Phi(p)\cdot\left[p^n+a_1p^{n-1}+\ldots+a_n+\sum_{m=0}^{l}k_m(p)p^m\right]=F(p).$$

From here will find a required function:

$$\Phi(p)=\frac{F(p)}{p^n+a_1p^{n-1}+\ldots+a_n+\sum_{m=0}^{l}k_m(p)p^m}.$$

The reconversion gives a required function.

1.13.6. Transformation Меллина

Let there is a certain function $f(t)$ and for it justly following:

$$\int_0^\infty |f(t)|t^{\sigma-1}dt<+\infty.$$

σ - arbitrary number:

Such function knows transformation Меллина:

$$F(S)=\int_0^{\infty} f(t)t^{S-1}dt.$$

Reconversion transformation

$$f(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(S)t^{-S}dS, \quad t>0.$$

The transformation Меллина establishes unambiguous interconnection between two by functions. The integral takes on a complex integrated plane on a vertical axis.

Example:

Let's consider a gamma-function. With the help of transformation Меллина.

$$\Gamma(S)=\int_0^{+\infty} e^{-t}t^{S-1}dt;$$

$$e^{-t}=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \Gamma(S)t^{-S}dS; \ \text{c}>0.$$

Transformation Меллина in many respects similar on a Laplace transformation:

$$\Phi(p)\xrightarrow{\text{Laplace transformation}}\varphi(t);$$

$$\Updownarrow \ p=S;$$

$$\Phi(S)\longleftarrow f(t).$$

In this case between functions *φ(t)* and *f(t)* there is an interconnection: $\varphi(t)=f\left(e^{-t}\right).$

1.13.7. Application of transformation Меллина for the solution of integral equations

Using convolution:

$$M\left\{\int_0^{\infty} f(t)\varphi\left(\frac{x}{t}\right)\frac{dt}{t}\right\}=F(S)\Phi(S);$$

$$F(S) = M\{f(t)\};$$
$$\Phi(S) = M\{\varphi(t)\}.$$

This property will be used for the solution of integral equations of a kind:

$$\varphi(x) = f(x) + \int_0^{+\infty} K(\frac{x}{t})\varphi(t)\frac{dt}{t}. \qquad (1.13.1)$$

The condition of applicability is, that the functions should admit transformation Меллина.

Let's designate transformation Меллина from $f(x)$ through $M\{f(x)\} = F(S)$, and transformation Меллина from $K(z)$ as $M\{K(z)\} = K(S)$.

Functions $F(S)$ and $K(S)$ should have general area of an analyticity.

Using transformation Меллина to both parts of an equation (1.13.1).

$$\Phi(S) = F(S) + K(S)\Phi(S);$$
$$\Phi(S) = \frac{F(S)}{1 - K(S)}.$$

By inverse of transformation we are finding *φ(t)*.

Example:

Let there is an integral equation of a kind:

$$\varphi(x) = e^{-\alpha x} + \frac{1}{2}\int_{-\infty}^{+\infty} e^{-\frac{x}{t}}\varphi(t)\frac{dt}{t}, \quad \alpha > 0.$$

Let's find transformation Меллина separately from each component:

$$M\{e^{-\alpha x}\} = \int_0^{+\infty} e^{-\alpha x}x^{S-1}dx = \alpha^{-S}\int_0^{+\infty} e^{-z}z^{S-1}dz = \frac{\Gamma(S)}{\alpha^S} = F(S), \quad \mathrm{Re}\{S\} > 0.$$

Replace:

$$M\left\{\frac{1}{2}e^{-x}\right\} = \frac{1}{2}\Gamma(S) = K(S); \quad \mathrm{Re}\{S\} > 0.$$

The areas of an analyticity coincide:

$$\Phi(S) = \frac{\Gamma(S)}{\alpha^S} + \frac{1}{2}\Gamma(S)\Phi(S);$$

$$\Phi(S) = \frac{\Gamma(S)}{\alpha^S \left[1 - \frac{1}{2}\Gamma(S)\right]}.$$

By inverse of transformation (Меллина) we are finding *φ(t)*.

1.14. Symmetrical integral equations

Symmetrical the integral equations are named for which kernels justly:

$$K(t,S) = K(S,t).$$

Example:

$$K(t,S) = t^2 S^2.$$

If a kernel complex- owes will be executed:

$$K(t,S) = K^*(S,t).$$

Let such function is a kernel of an equation:

$$\varphi(t) = \lambda \int_a^b K(t,S)\varphi(S)dS + f(t).$$

$A\varphi = \int_a^b K(t,S)\varphi(S)dS$ - The linear operator under a function φ.

If $f(t) \equiv 0$, that respective integral equation would become uniformly

Thus it is possible to write down following : $\varphi = \lambda A \varphi$. Such uniformly equation has restricted number of the solutions . These solutions represent a set of some functions $\{\varphi_C\}$. They are named as eigenfunctions and correspond to own numbers of a kernel- defined values λ.

The symmetrical kernel should have the next characteristics:

1. Every kernel should have minimum nonzero own number. And all own numbers are real.
2. To each own number there can correspond some eigenfunctions.

3. The eigenfunctions from different sets always are orthogonal among themselves, though inside a set, they are optionally orthogonal.

In each set quantity of functions nф can be estimated from a following inequality:

$$n_{\phi} \le \lambda^2 \int_a^b \int_a^b \left| K(t,S) \right|^2 dt dS .$$

For their further using they are necessary for orthogonalizing. For all that will be used the procedure of orthogonalization Грама-Шмидта.

At the first stage there are own numbers and eigenfunctions of an equation:

$$\varphi = \lambda A \varphi .$$

At the second stage achieve, that the inside sets of a function among themselves too orthogonal.

The functions $A(t), B(t)$ are called orthogonal, if:

$$\int_a^b A(t)B(t)dt = 0 .$$

The functions from a set appropriate to each own number subject to a procedure of orthogonalization. The procedure consists of several stages:

1. Choosing the first function, $\psi_1 = \varphi_{i1}(t)$:

$$\omega_1(t) = \frac{\psi_1(t)}{\sqrt{\int_a^b \psi_1^2(t)dt}} .$$

2. $\psi_2 = \varphi_{i2}(t) - \omega_1(t)\int_a^b \omega_1(t)\varphi_{i2}(t)dt$.

$$\omega_2(t) = \frac{\psi_2(t)}{\sqrt{\int_a^b \psi_2^2(t)dt}} .$$

We are finding the seconded function $\omega_2(t)$ from a new set.

3. $\psi_3 = \varphi_{i3}(t) - \omega_1(t)\int_a^b \omega_i(t)\varphi_{i3}(t)dt - \omega_2\int_a^b \omega_2\varphi_{i3}dt$;

$$\omega_3(t) = \frac{\psi_3(t)}{\sqrt{\int_a^b \psi_3^2(t)dt}} \to \omega_3(t).$$

We are finding the seconded function $\omega_3(t)$ from another set.

$$\psi_k = \varphi_{ik}(t) - \omega_1(t)\int_a^b \omega_i(t)\varphi_{ik}(t)dt - \omega_2\int_a^b \omega_2\varphi_{ik}dt - \omega_{k-1}\int_a^b \omega_{k-1}\varphi_{ik-1}dt;$$

$$\omega_k(t) = \frac{\psi_k(t)}{\sqrt{\int_a^b \psi_k^2(t)dt}}.$$

We will continue for the moment finding the last function from this set.

For finding the function justly:

$\int_a^b \omega_i\omega_j dt = 0$ - conditions orthogonal property.

$\int_a^b \omega_i^2 dt = 1$ - normalized conditions.

For improvements we will use the next exchange:

($\lambda \neq 0$).

Let's divide an input equation on λ, we shall designate $\mu = 1/\lambda$, and also $g(t) = -1/\lambda\, f(t)$.

Our integral equation will following kind:

$$\int_a^b k(t,S)\varphi(S)dS - \mu\varphi(t) = g(t).$$

Or in the statement form:

$$A\varphi - \mu\varphi = g.$$

1.15. Integral equations, which can be led to symmetrical

Some integral equations can be led to symmetrical and used for further computations.

For example, the equation:

$$\varphi(t) = \lambda \int_a^b k(t,S) p(S) \varphi(S) dS + f(t),$$

where *k* is a real symmetrical kernel.

It is assumed that the function $p(S) > 0$ in $[a,b]$. By multiplying both of the parts by $\sqrt{p(t)}$ and introducing a designation:

$$L(t,S) = k(t,S)\sqrt{p(t)p(S)}; \psi(t) = \sqrt{p(t)}\varphi(t);$$

$$\Psi(t) = \lambda \int_a^b L(t,S) \Psi(S) dS + \sqrt{p(t)} f(t).$$

This is a standard form of integral equation with symmetrical kernel. $\Psi(t)$ and later $\varphi(t)$ can be found by solving it.

1.16. 1st kind Volterra equations

$$\int_a^t K(t,S)\varphi(S)dS = f(t).$$

Differentiability of all functions in equation is assumed. It is necessary for continuous solution that $f(a) = 0$. To solve the equation need to compute the derivative of the function with respect to *t*. The result is:

$$K(t,t)\varphi(t) + \int_a^t \frac{\partial}{\partial t}[K(t,S)]\varphi(S)dS = \frac{\partial}{\partial t} f(t).$$

Assumed, that $K(t,t) \neq 0$.

Designations are as follows: $\frac{\partial}{\partial t} K(t,S) = K_t^{'}(t,S)$ and $\frac{\partial}{\partial t} f(t) = f_t^{'}(t)$;

$$\varphi(t)+\int_a^t \frac{K_t'(t,S)}{K(t,t)}\varphi(S)dS=\frac{f_t'(t)}{K(t,t)}.$$

The result is a second kind equation, what can be solved using regular ways. It could happen, that $K(t,t)\equiv 0$, then the computation yields to the second kind equation again.

$$\int_a^t \frac{\partial}{\partial t}\left[K(t,S)\right]\varphi(S)dS=\frac{d}{dt}f(t).$$

In such a case both parts are differentiable with respect to *t*.

$$K_t'(t,t)\varphi(t)+\int_a^t \frac{\partial^2 K(t,S)}{\partial t^2}\varphi(S)dS=f_t''(t).$$

If $K_t'(t,t)\neq 0$, then division by $K'(t,t)$ is leading to the second kind equation. If $K'(t,t)$ =0 anyway, the procedure has to be repeated again.

1.17. 1st kind Fredholm equations with symmetrical kernel

$$\int_a^b k(t,S)\varphi(S)dS=f(t). \qquad (1.17.1)$$

First kind Fredholm equation might have no solutions even in case of a "good" kernel. For instance, suppose the kernel be a power function with a finite number of terms:

$$k(t,S)=a_0(S)t^m+a_1(S)t^{m-1}+...+a_m(S).$$

It is easy to show that after a substitution into an integral it yields:

$$t^m\int_a^b a_0(S)\varphi(S)dS+t^{m-1}\int_a^b a_1(S)\varphi(S)dS+...+\int_a^b a_m(S)\varphi(S)dS=$$

$=t^m b_0+t^{m-1}b_1+...+b_m$ is a power function with a finite number of terms.

So, if for instance $f(t)=\sin t$, then left part will never yield to sin *t* with any coefficients if the number of *m* is finite. Therefore such an equation has no solution.

It is possible to try out to find a solution for symmetrical kernels using the Hilbert-Schmidt theorem. Then it is required that *f(t)* can be decomposed with respect to eigenfunctions of the kernel, i.e.

$$f(t)=\sum_i a_i \varphi_{ic}(t), \qquad (1.17.2)$$

where coefficients are: $\quad a_i = \int_a^b f(t)\varphi_{ic}(t)dt.$

Hilbert and Schmidt suggested to find a solution in the form of decomposed kernel's eigenfunctions but with another coefficients.

$$\varphi(t)=\sum_i c_i \varphi_{ic}(t). \qquad (1.17.3)$$

If the equation (1.17.3) is substituted into a first kind Fredholm equation (1.17.1) and compared to (1.17.2) then it yields to:

$\frac{c_i}{\lambda_i} = a_i$, where λ_i are self-numbers.

So, the final solution:

$$\varphi(t)=\sum_i a_i \lambda_i \varphi_{ic}(t).$$

1.18. Usage of a sequential approximation method to solve some of the first kind Fredholm's integral equations

Let $\lambda_{\min}$ be a minimal absolute eigenvalue of the kernel *K(t,S)*. If $0<|\lambda|<2|\lambda_{\min}|$, then a solution could be found as an iteration procedure in the following form:

$$\varphi(t)=\lim_{n\to\infty} \varphi_n(t),$$

where:

$$\varphi_n(t)=\varphi_{n-1}(t)+\lambda\left[f(t)-\int_a^b k(t,S)\varphi_{n-1}(S)dS\right].$$

Starting function $\varphi_0(t)$ can be taken as optional.

In this case the solving algorithm is represented as follows:

1) $k(t,S) \rightarrow .\{\ \lambda_i\ \} \rightarrow \lambda_{\min}$;
2) Selection of $\varphi_0(t)$ and λ;
3) Computing iterations.

1.19. Execute function method

Assumed, that the kernel *k* is symmetrical. Besides, it should be one of the generating functions. Function $G(t,z)$ is known as generating function for a system of initial functions $g_i(z)$, i.e.

$$G(t,z) \leftarrow \{ \ g_0(z), g_1(z), \ldots \ \},$$

if it can be represented as follows:

$$G(t,z) = \sum_{n=0}^{\infty} C_n g_n(z) t^n .$$

Each of the functions *g(z)* is orthogonal:

$$\int_a^b g_i(z) g_j(z) dz = 0, \quad i \neq j .$$

The solution could be found in the form of:

$$\varphi(t) = \sum_{n=0}^{\infty} a_n g_n(t). \tag{1.19.1}$$

After the substitution of the kernel and the sought function into the integral and transformation:

$$\int_a^b k(t,S)\varphi(S)dS = \int_a^b \sum_{n=0}^{\infty} C_n g_n(S) t^n \left[\sum_{k=0}^{\infty} a_k g_k(S) \right] dS =$$

$$= \sum_{n=0}^{\infty} C_n t^n \left[\sum_{k=0}^{\infty} a_k \int_a^b g_n(S) g_k(S) dS \right] = \sum_{n=0}^{\infty} C_n t^n a_n G_n ,$$

where $G_n = \int_a^b g_n^2(S) dS$.

By computing the derivative of the function $f(t)$ *k* times and substituting *t=0*, a_n can be found:

$$\left. \frac{\partial^k f(t)}{\partial t^k} \right|_{t=0} = C_k a_k G_k k!$$

There will be only one coefficient in decomposition, for instance:

$$y(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + ...$$
$$y'(t) = b_1 + 2b_2 t + 3b_3 t^2 + ... \Big|_{t=0} = b_1;$$
$$y''(t) = 2b_2$$
....

etc.

So,

$$a_k = \frac{f^{(k)}(t)\Big|_{t=0}}{C_k G_k k!}.$$

After substituting these coefficients into (1.19.1), it will be like:

$$\varphi(t) = \sum_{k=1}^{\infty} \frac{f^{(k)}(t)\Big|_{t=0}}{G_k C_k k!} g_k(t).$$

1.20. Non-Fredholm integral equations

Kernels corresponding to the condition:

$$\int_a^b \int_a^b |k(t,S)|^2 \, dt dS < +\infty.$$

were overviewed before.

If this condition is false, then continuous areas of numbers and corresponding joint of continuous functions are conformed to the kernel, but not a set of eigenvalues as before.

For example (Picard equation):

$$\varphi(t) = \lambda \int_{-D}^{D} e^{-|t-S|} \varphi(S) dS.$$

Let's check if it is a Fredholm equation.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(t,S)|^2 \, dt dS = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-2|t-S|} dS \right] dt =$$
$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{t} e^{-2(t-S)} dS + \int_{t}^{\infty} e^{-2(S-t)} dS \right] dt = \int_{-\infty}^{\infty} dt \cdot 1.$$

It is not a Fredholm type of equation.

The solution for this equation is:

$$\varphi_1(t) = c_1 e^{rt} + c_2 e^{-rt};$$

$$r = \sqrt{1-2\lambda}\,,\ \lambda > 0\,.$$

The collection of eigenvalues forms a continuous set.

Have a look at the dual couple of functions: continuous and integratable function *φ(t)* and its cosine transform $\varphi_1(\omega)$:

$$\varphi_1(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \varphi(x) \cos \omega x dx;$$

$$\varphi(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \varphi_1(\omega) \cos \omega x dx.$$

After forming a function *ψ(x)* out of them and proceeding to other variables:

$$\psi(x) = \varphi(x) + \varphi_1(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left[\varphi_1(t) + \varphi(t)\right] \cos \omega x dt = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \psi(t) \cos xt dt.$$

The function *ψ(x)* is an integral equation's eigenfunction:

$$\psi(t) = \lambda \int_0^{\infty} \psi(t) \cos xt dt,$$

and it corresponds to the eigenvalue:

$$\lambda = \sqrt{\frac{2}{\pi}}.$$

As *ψ(x)* might be optional, so if $\lambda = \sqrt{\frac{2}{\pi}}$ then the integral equation has an infinite number of eigenfunctions. This case could take place as:

$$\int_0^{\infty}\int_0^{\infty} |K|^2 \, dx dt = \int_0^{\infty}\int_0^{\infty} \cos^2 x \, dx dt = \infty.$$

1.21. Singular integral equations

The singular integral equation is known as integral equation where an unknown function stands under the singular integral.

Assume the function $f(x)$ is unlimited at the neighbourhood of x_0 ($f \to \infty$ if $x \to x_0$).

Cauchy's main value is known as a limit (if it exists):

$$\lim_{\xi \to 0} \left[\int_{a}^{x_0 - \xi} f(x)\,dx + \int_{x_0 + \xi}^{b} f(x)\,dx \right].$$

It means that $0<\xi<\min\{x_0 - a; b - x_0\}$.

All integrals in the meaning of this value are known as special or singular integrals.

Designation that is used for them is:

$$V.p.\int_{a}^{b} f(x)dx.$$

For instance:

$$\int_{a}^{b} \frac{dx}{x-c}, c \in [a,b].$$

We had been discussing the use of such integrals earlier: integral equations with weak peculiarity kernel.

Now, let's have a look at the important case from radioengineering point of view.

1.22. Hilbert transform

The integral Fourier transform:

$$f(x) = \int_{0}^{+\infty} [a(t)\cos xt + b(t)\sin xt]dt.$$

The coefficients can be determined using the following formulas:

$$a(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(u)\cos ut du;$$

$$b(t)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(u)\sin utdu\,.$$

Integral Fourier transform can be considered as a limit (when $y\to 0$) of the expression $\lim\limits_{y\to 0} U(x,y),$ where:

$$U(x,y)=\int_{0}^{\infty}\left[a(t)\cos xt+b(t)\sin xt\right]e^{-yt}dt.$$

This integral could be treated as a real part of more complicated one:

$$\Phi(z)=\int_{0}^{\infty}\left[a(t)-jb(t)\right]e^{jxt-yt}dt=\int_{0}^{\infty}\left[a(t)-jb(t)\right]\left[\cos xt+j\sin xt\right]e^{-yt}dt=$$

$$=\int_{0}^{\infty}\left[a(t)\cos xt+b(t)\sin xt\right]e^{-yt}dt-j\int_{0}^{\infty}\left[b(t)\cos xt-a(t)\sin xt\right]e^{-yt}dt=$$

$$=U(x,y)+jV(x,y),$$

where $V(x,y)$ is an imaginary part of the complex function $\Phi(x,y)$.

The limit of the function $V(x,y)$ can be found as (if $y\to 0$):

$$g(x)=-V(x,0)=\int_{0}^{\infty}\left[b(t)\cos xt-a(t)\sin xt\right]dt\,.$$

The function $g(x)$ is expressed from $f(x)$. After a substitution:

$$g(x)=\frac{1}{\pi}\int_{0}^{\infty}dt\left\{\int_{0}^{\infty}f(u)\sin\left[(u-x)t\right]dx\right\}.$$

This integral is conjugated to Fourier integral transform.

By repeating the procedure we can get an initial expression but it will be negative:

$$f(x)=-\frac{1}{\pi}\int_{0}^{\infty}dt\left\{\int_{0}^{\infty}g(u)\sin\left[(u-x)t\right]du\right\}.$$

After a several formal transforms:

$$g(x)=\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{0}^{\lambda}dt\left\{\int_{-\infty}^{\infty}\sin\left[(u-x)t\right]f(u)du\right\}=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\left[\lim_{\lambda\to\infty}\int_{0}^{\lambda}\sin\left\{\left[(u-x)t\right]dt\right\}du\right]=$$

$$=\frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\infty}^{\infty}\frac{1-\cos\lambda(u-x)}{u-x}f(u)du.$$

The integral can be divided into two parts:

$$\int_{-\infty}^{\infty} \frac{1-\cos\lambda(u-x)}{u-x} f(u)du = \int_{-\infty}^{x} \frac{1-\cos\lambda(u-x)}{u-x} f(u)du +$$

$$+ \int_{x}^{\infty} \frac{1-\cos\lambda(u-x)}{u-x} f(u)du.$$

Then, substitute t by $u - x = t$. So, the second integral in a sum will be:

$$\int_{-\infty}^{x} \frac{1-\cos\lambda(u-x)}{u-x} f(u)du = \int_{-\infty}^{0} \frac{1-\cos\lambda t}{t} f(x+t)dt.$$

The first integral:

$$\int_{x}^{\infty} \frac{1-\cos\lambda(u-x)}{u-x} f(u)du = \int_{0}^{\infty} \frac{1-\cos\lambda t}{t} f(x+t)dt.$$

After changing the sign of the expression ($t = -t$):

$$\int_{+\infty}^{0} \frac{1-\cos\lambda(-t)}{-t} f(x-t)d(-t) = \int_{+\infty}^{0} \frac{1-\cos\lambda t}{t} f(x-t)dt =$$

$$= -\int_{0}^{\infty} \frac{1-\cos\lambda t}{t} f(x-t)dt.$$

Finally,

$$g(x) = \lim_{\lambda\to\infty} \frac{1}{\pi} \int_{0}^{\infty} \frac{1-\cos\lambda t}{t} [f(x+t) - f(x-t)]dt.$$

A part of the integral containing $\cos(\lambda t)$ approaches zero for considerably smooth functions $f(x)$ as proved by Hilbert.

So,

$$\begin{cases} g(x) = \dfrac{1}{\pi} \displaystyle\int_{0}^{\infty} \frac{f(x+t) - f(x-t)}{t} dt; \\ f(x) = -\dfrac{1}{\pi} \displaystyle\int_{0}^{\infty} \frac{g(x+t) - g(x-t)}{t} dt. \end{cases}$$

This duality was noticed for the first time by Hilbert. So, two functions linked by such a transform are known as Hilbert transform.

More often they are used in another form:

$$g(x) = \frac{1}{\pi} V.p. \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt ;$$

$$f(x) = -\frac{1}{\pi} V.p. \int_{-\infty}^{\infty} \frac{g(t)}{t-x} dt .$$

1.23. Usage of Hilbert transform for integral equations solving

Each of two Hilbert formulas can be considered as a first kind integral equation. Then another formulae will be a solution to that integral equation.

Let:

$$f(x) = \mathrm{H}[\varphi(t)] = \frac{1}{\pi} V.p. \int_{-\infty}^{+\infty} \frac{\varphi(y)}{y-x} dy .$$

Be a Hilbert transform of the function φ. This method is used to solve equations in the form of:

$$\varphi(x) - \lambda \int_{-\infty}^{+\infty} \frac{\varphi(y)}{y-x} dy = f(x).$$

Remembering upper designation the symbolic form will be as follows:

$$\varphi(x) - \lambda\pi\mathrm{H}[\varphi] = f(x). \qquad (1.23.1)$$

By applying the Hilbert transform to the both sides of the equation we will get:

$$\mathrm{H}[\varphi] + \lambda\pi\varphi = \mathrm{H}[f];$$

(Consider that $\mathrm{H}\{\mathrm{H}[\varphi]\} = -\varphi$.)

By substituting $\mathrm{H}[\varphi]$ into (1.23.1):

$$\varphi - \lambda\pi\mathrm{H}[f] + \lambda^2\pi^2\varphi = f(x).$$

So, φ will be:

$$\varphi(1 + \lambda^2\pi^2) = f(x) + \lambda\pi\mathrm{H}[f];$$

$$\varphi(x) = \frac{f(x) + \lambda\pi\mathrm{H}[f]}{1 + \lambda^2\pi^2}; \qquad 1 + \lambda^2\pi^2 \neq 0.$$

The Hilbert transform is applicable in more complicated cases also, when kernel looks like:

$$k(x,y)=\frac{1}{y-x}+k_0(x,y);$$

Sometimes the Hilbert transform is used in the form of:

$$\begin{cases}\Psi(x)=\frac{1}{2\pi}V.p.\int\limits_{-\pi}^{+\pi}\varphi(t)ctg\left(\frac{t-x}{2}\right)dt;\\ \varphi(x)=-\frac{1}{2\pi}V.p.\int\limits_{-\pi}^{+\pi}\Psi(t)ctg\left(\frac{t-x}{2}\right)dt.\end{cases}$$

They are used if there is a *ctg* function in the equation. The way of solving is the same.

1.24. Nonlinear integral equations

Solving nonlinear integral equations is much difficult. Solution of the integral equation in the form of:

$$\varphi(t)=\lambda\int\limits_a^b k[t,S,\varphi(S)]dS+f(t),$$

was discussed earlier.

Consider the Gammerstein integral equation:

$$\varphi(t)=\lambda\int\limits_a^b k(t,S)\Psi[S,\varphi(S)]dS+f(t),$$

where $k(t,S)$, $\Psi(s,z)$ are known functions,

$\varphi(s)$ is sought function.

Condition:

$$\left|\frac{\partial\,\Psi(x,y)}{\partial y}\right|<|\lambda|_{\min}$$

has to be true. It is a minimal absolute value of the kernel's $k(t,s)$ eigenvalue.

The solution for this integral equation also can be found using the sequential approximation method. The function φ_0 is optional, even it is possible to construct an approximation series:

$$\varphi_1(x) = \lambda \int_a^b k(x,S)\Psi\left[S,\varphi_0(S)\right]dS + f(x);$$

$$\varphi_2(x) = \lambda \int_a^b k(x,S)\Psi\left[S,\varphi_1(S)\right]dS + f(x);$$

$$\varphi_3(x) = \lambda \int_a^b k(x,S)\Psi\left[S,\varphi_2(S)\right]dS + f(x);$$

. . . .

$$\varphi_n(x) = \lambda \int_a^b k(x,S)\Psi\left[S,\varphi_{n-1}(S)\right]dS + f(x);$$

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x).$$

1.25. Usage of degenerated kernels for Gammerstein equation solving

If the kernel is degenerated then it could be represented as follows:

$$k(t,S) = \sum_{i=1}^{m} a_i(t) b_i(S),$$

so, the initial integral equation in this case looks like:

$$\varphi(t) = \lambda \sum_{i=1}^{m} a_i(t) \int_a^b b_i(S)\Psi[S,\varphi(S)]dS + f(t).$$

and is known as Gammerstein equation.

Designating:

$$\varphi(t) = \lambda \sum_{i=1}^{m} a_i(t) C_i + f(t), \tag{1.25.1}$$

where: $C_i = \int_a^b b_i(S)\Psi[S,\varphi(S)]dS.$

By substituting (1.25.1) into the initial equation:

$$\int_a^b b_j(S)\psi(S, \sum_{i=1}^m c_i a_i(S) + f(S))dS = c_j\,,\ j = 1 \div m\,.$$

Functions $\Psi(\ )$ are known, so the integral can be solved. If the solution of the new formed nonlinear algebraic equation system exists, it means that a set of coefficients $\left\{c_1^\Delta \div c_m^\Delta\right\}$ exists also. By substituting them into a corresponding equation it can be converted into a true identity $\varphi(t) \equiv \sum_{i=1}^m c_i^\Delta a_i(t) + f(t)$. It could be, that there are not only one set of the coefficients $\left\{c_1^\Delta \div c_m^\Delta\right\}$ so, in this case we have several solutions $\varphi(t)$ for the integral equation.

Examples.

1. The initial integral equation in the form of:

$$\varphi(t) = \lambda\int_0^1 t^2 S\varphi^2(t)dt\,,$$

where $K(t,S) = t^2 S$ is a degenerated kernel consisted of one member. So, there is only one coefficient c exists.

$$c = \int_0^1 S\varphi^2(S)dS;$$

$$\varphi(t) = \lambda c t^2;$$

$$c = \int_0^1 c^2\lambda^2 S^4(S)dS = \frac{c^2\lambda^2}{6}.$$

Easy to notice that this algebraic equation has two solutions ($\lambda \neq 0$).

$$c_1 = 0;$$

$$c_2 = \frac{6}{\lambda^2}.$$

So, the integral equation has two solutions also:

$$\varphi_1(t) \equiv 0;$$

$$\varphi_2(t) = \frac{6}{\lambda^2}.$$

2. The initial equation:

$$\varphi(t) = \int_0^1 a(t)a(S)\varphi(S)\sin\left(\frac{\varphi(S)}{a(S)}\right)dS.$$

($a(t) > 0$ in interval t from 0 to 1).

By computing similar transforms, the equations with respect to coefficient c can be found.

$$1 = \int_0^1 a^2(S)\sin c dS;$$

$$1 = \sin c \int_0^1 a^2(S)dS.$$

Two variants are possible:

a) $\int_0^1 a^2(S)dS < 1$ — the solution does not exist.

b) $\int_0^1 a^2(S)dS > 1$.

then:

$$\sin c = \frac{1}{\int_0^1 a^2(S)dS};$$

$$c = \left\{ \begin{array}{l} \arcsin c = \dfrac{1}{\int_0^1 a^2(S)dS} + 2\pi n \\ \pi - \arcsin \dfrac{1}{\int_0^1 a^2(S)dS} + 2\pi n \end{array} \right\}.$$

There is an infinite number of c, so there is an infinite number of $\varphi(t)$ also.

3. The initial equation:

$$\varphi(t) = 1 + \lambda \int_0^1 \varphi^2(S)dS.$$

Assume: $$c = \int_0^1 \varphi^2(S)dS;$$

then: $$\varphi(t) = 1 + \lambda c.$$

Substituting it into the initial equation results:

$$\lambda^2 c^2 + (2\lambda - 1)c + 1 = 0.$$

Expression to find *c*:

$$c = \frac{1 - 2\lambda \pm \sqrt{1 - 4\lambda}}{2\lambda^2}.$$

Finally:

$$\varphi_1(t) = \frac{1 + \sqrt{1 - 4\lambda}}{2\lambda};$$

$$\varphi_2(t) = \frac{1 - \sqrt{1 - 4\lambda}}{2\lambda}.$$

It is possible for the Gammerstein equation with nondegenerated kernel to find a degenerated kernel that will approximate nondegenerated kernel in an integration interval rather precisely. In this case the solution of integral equation with degenerated kernel is an approximate solution of the integral equation with nondegenerated kernel.

2. CALCULUS OF VARIATIONS

Calculus of variations is the field of mathematics researching extremums of functions and functionals. If a solutions of extremum is found with any conditions, such problems is named as conditional.

2.1. The finding of function extremums

Let necessary to find a function extremum:

$$Z = f(x_1,...,x_n).$$

Available additional conditions require formalization, i.e. transformation to a set of functions with respect to *x* represented as:

$$\left\{\begin{matrix} \varphi_1(x_1,...,x_n) = 0 \\ \varphi_2(x_1,...,x_n) = 0 \\ --------- \\ \varphi_m(x_1,...,x_n) = 0 \end{matrix}\right\},$$

on conditions that *m*<*n*.

This is a default target setting. Such problems are solved using a couple of methods.

Method 1.

1. From one (any) connection equation, one of variables is expressed $x_1 \leftarrow \varphi_1(x_1,...,x_n)$.
Acquired x_1 is substituted in *f* and in $\varphi_2...\varphi_m$.
2. From another equation, $x_2 \leftarrow \varphi_2$ is expressed and substituted in f, $\varphi_3...\varphi_m$, etc. The same way repeating *m* times. $f(x_{m+1},...,x_n)$ is aquired depeding on *n-m* arguments, and the conditions are none.
3. Find conditionless extremum of *f* and substitute in connection equation with reverse order.

2.2. The method of lagrange multiplier for finding function extremums

Application condition:

1. Functions $f(x_1 \div x_n)$ and $\varphi(x_1 \div x_m)$ must be continuous and its partial derivatives must exist on all arguments.

2. In whole range of definition x, rank of matrix with elements $\frac{d\varphi_i}{dx_j}$ and size [n×m] must be greater than *m*.

On method using:

a) compose Lagrange function represented as:

$$\Phi = f + \sum_{i=1}^{m} \lambda_i \varphi_i ,$$

where λ_i are undefined Lagrange multipliers (unknown coefficients).

b) compose *n* equations represented as:

$$\begin{cases} \frac{\partial \Phi}{\partial x_1} = 0; \\ \frac{\partial \Phi}{\partial x_2} = 0; \\ \dots\dots\dots \\ \frac{\partial \Phi}{\partial x_n} = 0. \end{cases}$$

and *m* equations represented as:

$$\begin{cases} \varphi_1(x_1 \dots x_n) = 0; \\ \varphi_2(x_1 \dots x_n) = 0; \\ \dots \dots \dots \dots \dots \dots \\ \varphi_m(x_1 \dots x_n) = 0. \end{cases}$$

Thus, we have *m*+*n* equations and *m*+*n* variables. Then, solve this system. Points, at these a derivative of function *f* on all arguments $x_1 \dots x_n$ is equal to zero, are named stationary points. If only single solution exists, then corresponding extremum is said as a global extremum. If several solutions exist, then the function has several local extremums. Aquired sets *x* point to extremum coordinates. After then, necessary check each extremum. At that three case are possible at each point:

1) Maximum.
2) Minimum.
3) Saddle point.

The checking is processed as the following:

Compose quadratic form:

$$d^2\Phi = \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_k}\Delta x_i \Delta x_k \Bigg|_{\substack{x_1 = x_1^\nabla \\ x_2 = x_2^\nabla \\ \dots\dots\dots \\ x_n = x_n^\nabla}} \quad (x_1^\nabla \div x_n^\nabla) \text{ - system solution}$$

If at some small neighborhood is:

$d^2\Phi > 0$ - then there is a maximum;

$d^2\Phi < 0$ - then there is a minimum.

If $d^2\Phi$ may be greater than zero and less than zero, then there is a saddle point.

<u>Example 1.</u>

A function is given:

$$z = (x-1)^2 + (y+1)^2.$$

Find extremum with the following condition:

$$\varphi(x, y) = x + y - 1 = 0 .$$

In the first, we are finding a unconditional extremum:

$$\frac{\partial z}{\partial x} = 2(x-1) = 0; \quad x^\nabla = 1;$$

$$\frac{\partial z}{\partial y} = 2(y+1) = 0; \quad y^\nabla = -1;$$

$$z_{ex} = z_{\min} = 0.$$

Now, find a conditional extremum.

Express y from the condition:

$$y = 1 - x;$$

$$z = (x-1)^2 + (2-x)^2;$$

$$\frac{\partial z}{\partial x} = 2(x-1) + 2(2-x) = 0; \quad x^{\nabla} = 1{,}5; \quad y^{\nabla} = -0{,}5;$$

$$z_{\min} = 1/2.$$

The presence of the condition leads to other value of extremum and to other coordinate of extremum.

<u>Example 2.</u> (Lagrange multiplier method).

$$f(x,y,z) = xyz;$$

$$\varphi_1(x,y,z) = x + y - z - 3 = 0;$$

$$\varphi_2(x,y,z) = x - y - z - 8 = 0;$$

Compose Lagrange function:

$$\Phi(x,y,z) = xyz + \lambda_1(x + y - z - 3) + \lambda_2(x - y - z - 8);$$

$$\begin{cases} \dfrac{\partial \Phi}{\partial x} = yz + \lambda_1 + \lambda_2 = 0; \\ \dfrac{\partial \Phi}{\partial y} = xz + \lambda_1 - \lambda_2 = 0; \\ \dfrac{\partial \Phi}{\partial z} = xy - \lambda_1 - \lambda_2 = 0; \\ x + y - z - 3 = 0; \\ x - y - z - 8 = 0; \end{cases}$$

Solving that, we get:

$$\lambda_1 = \frac{11}{32}; \ \lambda_2 = -\frac{231}{32}; \ x^{\nabla} = \frac{11}{4}; \ y^{\nabla} = -\frac{5}{2}; \ z^{\nabla} = -\frac{11}{4}; \ f_{ex} = \frac{605}{32}.$$

Determine a kind of the found point.

$$\frac{\partial^2\Phi}{\partial x^2} = \frac{\partial^2\Phi}{\partial y^2} = \frac{\partial^2\Phi}{\partial z^2} = 0;$$

$$\frac{\partial^2\Phi}{\partial x\partial y} = z; \quad \frac{\partial^2\Phi}{\partial x\partial z} = y; \quad \frac{\partial^2\Phi}{\partial y\partial z} = x;$$

$$d^2\Phi = 2xdydz + 2ydxdy + 2zdxdy\ .$$

From connection condition:

$$\begin{cases} dx + dy - dz = 0; \\ dx - dy - dz = 0; \end{cases}$$

$$dy = 0; \quad dx = dz;$$

$$d^2\Phi = 2y^{\nabla} dx^2.$$

$$d^2\Phi = 2\left(-\frac{5}{2}\right)dx^2 = -5dx^2 < 0.$$

The quadratic form is greater than zero regardless to the sign of *x,* that means a maximum placed at the investigated point.

2.3. Functional

Assume, some class *M* of functions *y(x)* is given. If each function $y(x) \in M$ is accordance to some number *J* by some rule, then it is said a functional *J* is defined in the class *M*.

$$J = J[y(x)].$$

The class *M*, where this functional is defined, is named as a domain of functional.

Example 1.

Assume that *M* is a collection of all continuous function at range [0,1]. Following define integral is a functional:

$$J[y(x)] = \int_0^1 y(x)dx\,;$$

when: $y(x) = c \;\rightarrow\; J = c;$

when: $y(x) = e^{x} \;\rightarrow\; J = e - 1;$

when: $y(x) = \cos \pi x \;\rightarrow\; J = 0.$

Example 2.

Assume that *M* is a class of functions having a continuous derivative at range $[a,b]$ and let $x_0 \in [a,b]$, then the following is considered as a functional:

$$J = y'(x_0); \quad a = 1; \quad b = 3; \quad x_0 = 2;$$

when: $y(x) = x^2 \;\rightarrow\; J = 4;$

when: $y(x) = \ln(1 + x) \;\rightarrow\; J = \frac{1}{3}.$

2.4. Variations

A variation (increment) of δy being a argument of $y(x)$ of a functional $J[y(x)]$ is a difference between a couple of functional, when both functions are included to class *M*.

$$\delta y = y(x) - y(x_0).$$

If the function *y* can be derivatived *k*-times, then degree of the variation is *k*.

$$(\delta y)^{(k)} = \delta y^{(k)}(x) = y^{(k)}(x) - y^{(k)}(x_0).$$

It's said that functions $y(x)$ and $y_1(x)$ are close in terms of zero order, if the condition, that $|y(x) - y_1(x)|$ is small, is satisfied. Geometrically it means that at this range the functions are close by arguments. There is the closeness of first order, if not only a difference between theirs is small, but a difference between their derivations is small too.

$$\begin{cases} |y(x) - y_1(x)|; \\ |y'(x) - y'(x)|. \end{cases} \text{ - small.}$$

The closeness of *k*-th order – the condition is added:

$$\left|y^{(K)}(x)-y_1^{(K)}(x)\right| \text{ - small.}$$

(and all differences of lower orders are small too).

If the closeness of *k*-th order is given, then there is the closeness of previous order.

Example.

There are the curves $y(x)=\frac{\sin^2}{n}$ and $y_1(x)\equiv 0$. Consider theirs ar range $[0,\pi]$. We can claim these are close in term of zero order when n are great.

$$\left|y(x)-y_1(x)\right|=\left|\frac{\sin n^2 x}{n}\right| \xrightarrow[n\to\infty]{} 0.$$

In terms of first order there is not the closeness, because at point:

$$x=\frac{2\pi}{n^2};$$

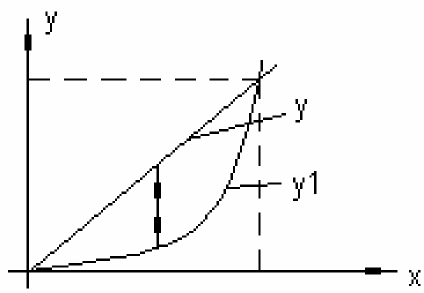
$$\left|y'(x)-y_1'(x)\right|=n\left|\cos n^2 x\right|,$$

this expression can be maked arbitrary large by *n* growing.

Distance between curves $y=f(x)$, $y_1=f_1(x)$ at range $a\div b$ (consider both functions as continuous) is a positive number ρ, which is equal maximum modulus of difference between them.

Example.

There are the functions $y=x$ and $y_1=x^2$; $a\div b=0\div 1$.



$$\rho_1(x) = y - y_1 = x - x^2;$$
$$\frac{d\rho_1}{dx} = 1 - 2x;$$
$$1 - 2x = 0.$$

Maximal distance at point:

$$x = \frac{1}{2},$$

it is eual to $$\rho = \frac{1}{4}.$$

Distance of *n*-th order between curves is the most of maxima of following values:

$$\left|f(x) - f_1(x)\right|;$$
$$\left|f^{'}(x) - f_1^{'}(x)\right|;$$
$$\ldots\ldots\ldots\ldots$$
$$\left|f^{(n)}(x) - f_1^{(n)}(x)\right|,$$

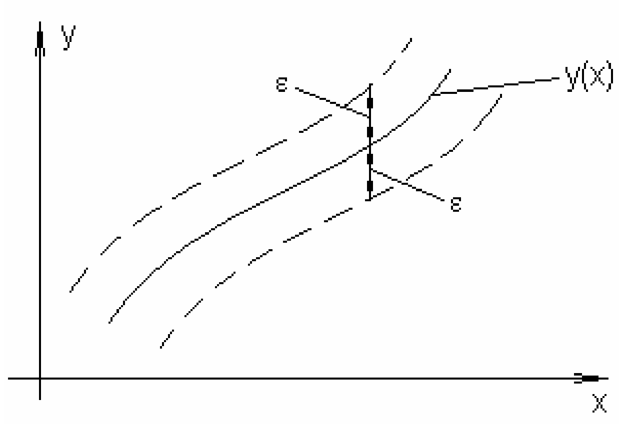
at range $[a,b]$.

$$\rho_n = \rho_n\left[f(x), f_1(x)\right] = \max_{0 \le k \le n} \max_{a \le x \le b} \left|f^{(k)}(x) - f_1^{(k)}(x)\right|.$$

ε neighborhood of *n*-th order of a curve $y(x)$ at range $[a,b]$ is a collection of curves $f_1(x)$, distance of *n*-th order from original curve $y(x)$ is less than ε.

$$\rho_n = \rho_n\left[y(x), f_1(x)\right] < \varepsilon.$$

Neighborhood of zero order is a strong neighborhood. Neighborhood of first order is a weak neighborhood. Physically meaning of a strong neighborhood is a set of continuous curves, which can be drawn in a belt with width 2ε about a curve *y=f(x)*.



A functional $J[y(x)]$ in M class of functions is continuous, when $y = y_0(x)$ in terms of the closeness of n-th order, if for any ε we can select such number $\eta > 0$ to satisfy condition:

If $\rho_n[y(x), y_0(x)] < \eta$; then $|J[y(x)] - J[y_0(x)]| < \varepsilon$.

In the other case, it is discontinuous. A functional is linear, if all properties of linear operators are right.

2.5. The simplest problem of calculus of variations

The functional is given:

$$J[y] = \int_{x1}^{x2} F(x, y, y^{'})dx,$$

where F is a unknown function,
y is a unknown piecewise-smooth function.

It needs to find minimum of this functional among all piecewise-smooth functions y.

Conditions:

1. Function $y(x)$ must connect points $y_1=y(x_1)$ and $y_2=y_2(x_2)$.

2. $F(x, y, y^{'})$ must be continuous in all three arguments (x, y, y'), and all derivatives must be continuous up to third order too.

Minimum (maximum) of functional $J[y]$, reached in a strong (weak) neighborhood of function $y_0(x)$ is named as a strong (weak) minimum (maximum) of functional $J[y]$. A extremum of functional $J[y]$ at whole set of functions *y,* where it defined, is a absolute extremum.

2.6. The required condition of extremum. First and second variation of functional

Assume $\eta(x)$ is a piecewise-smooth function, which is satisfied for the condition:

$$\eta(x_1) = \eta(x_2) = 0.$$

Introduce a function:

$$\tilde{y}(x) = y(x) + \alpha\eta(x),$$

where α is a unknown parameter.

Then a set of all possible functions $\tilde{y}(x)$ is owned by a weak neighborhood of function *y.*

Functional:

$$J[\tilde{y}] = \int_{x1}^{x2} F(x, \tilde{y}, y')dx.$$

by conditions:

$$\begin{bmatrix} \tilde{y}(x) = y(x_1) = y \\ \tilde{y}(x_2) = y(x_2) = y_2 \end{bmatrix}.$$

is a function of the parameter α.

$$J[\tilde{y}] = \int_{x1}^{x2} F(x, y + \alpha\eta, y' + \alpha\eta')dx = \Phi(\alpha).$$

Shown that $\Phi(\alpha)$ has minimum when $\alpha = 0$.

For this the following conditions are required:

$$\begin{cases} \dfrac{\partial\Phi(\alpha)}{\partial\alpha} = 0, \text{ when } \alpha = 0, \\ \dfrac{\partial^2\Phi(\alpha)}{\partial\alpha^2} \geq 0, \text{ when } \alpha = 0 \end{cases}.$$

After differenting $\Phi(\alpha)$ by the parameter α:

$$\frac{\partial \Phi}{\partial \alpha} = \int_{x1}^{x2} \left[\frac{\partial}{\partial y} F(x, \widetilde{y}, \widetilde{y}^{'})\eta + \frac{\partial}{\partial (y^{'})} F(x, \widetilde{y}, \widetilde{y}^{'})\eta \right] dx = 0 . \quad (2.6.1.)$$

A derivative $\frac{\partial \Phi}{\partial \alpha}$ at point $\alpha = 0$ is named as first variation of functional $J[y]$ and labeled as:

$$\delta J = \left. \frac{d\Phi}{d\alpha} \right|_{\alpha=0} .$$

Corresponding derivative is named as second variation.

$$\delta^2 J = \left. \frac{d^2\Phi}{d\alpha^2} \right|_{\alpha=0} .$$

Found function y give minumum (maximum) $J[y]$, if:

$$\begin{cases} \delta J = 0 \\ \delta^2 J > 0 \ - \text{ minimum} \\ \delta^2 J < 0 \ - \text{ maximum} \end{cases} .$$

If integrate expression (6.1.) by apart, then we'll get:

$$\delta J = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial (y')} - \int_{x_1}^{x} \frac{\partial F}{\partial y} dx \right] \eta' \, dx = 0 .$$

This expression must be satisfied for any η. From this the Euler-Lagrange integral equation is following:

$$F_{y'} - \int_{x_1}^{x} F_y dx \equiv C .$$

After differenting, we get:

$$\left(\frac{\partial F}{\partial (y')} = F_{y'} ; \ \frac{\partial F}{\partial y} = F_y ; \right)$$

$$F_y - \frac{d}{dx}\left(F_{y'}\right) = 0 . \quad (2.6.2.)$$

Expression (2.6.2.) is one of basic equation of calculus of variation. It is just as first finding extremum.

Smooth function *y(x)* being a solution of this equation is named as the extremal. The extremal is named as Lagrange curve also. The extremal is satisfied for (2.6.2.), also satisfied for the following equation:

$$\frac{d}{dx}\left[F - y' F_{y'}\right] - F_x = 0.$$

Besides a detailed representation is used:

$$y'' F_{y'y'} + y' F_{y'y} + F_{y'x} - F_y = 0;$$

$$F_{y'y'} = \frac{d}{d(y')}\left[\frac{d}{d(y')} F\right];$$

$$F_{y'y} = \frac{d}{dy}\left[\frac{d}{d(y')} F\right];$$

$$F(x, y, y') = F(X, Y, Z).$$

Although, arguments are connected each to other, but, while it is differenting with respect to one of arguments, others arguments are considered as constants.

Notes.

1. This formula gives solution to two constants, and these are determinated from boundary conditions.
2. For specified boundary conditions, equation has no solution or has infinite number of solutions.

<u>Example.</u>

1) The functional is given:

$$J[y(x)] = \int_1^2 \left[y'^2 - 2xy\right] dx, \text{ when } y(1) = 0,\ y(2) = -1;$$

$$F(x, y, y') = y'^2 - 2xy;$$

$$F_{y'} = \frac{d}{d(y')} F = 2y';$$

$$\frac{d}{dx}\left(F_{y'}\right) = \frac{d}{dx}(2y') = 2y''; \; F_y = \frac{\partial F}{\partial y} = -2x; \; -2x - 2y'' = 0; \; y + x = 0;$$

$$y = -\frac{x^3}{6} + C_1 x + C_2.$$

Substitute boundary conditions:

$$\begin{cases} C_1 + C_2 = \dfrac{1}{6}; \\ 2C_1 + C_2 = \dfrac{2}{6}; \end{cases}$$

$$C_1 = \frac{1}{6}; \; C_2 = 0; \; y = \frac{x}{6}\left(1 - x^2\right).$$

2) Find extremum of the functional:

$$J[y(x)] = \int_1^3 (3x - y) y dx.$$

Boundary conditions:

$$y(1) = 1; \; y(3) = 4{,}5.$$

Euler equation is as the following:

$$3x - 2y = 0;$$

$$y(x) = 1{,}5x.$$

Aquired extremal isn't satisfied the first boundary condition. It means that the problem could not be solved.

Find a extremal of following functional:

$$J[y(x)] = \int_0^{2\pi} \left(y'^2 - y^2\right) dx.$$

$$y(0) = 1; \; y(2\pi) = 1.$$

Euler equation is:

$$y'' + y = 0.$$

General solution is:

$$y(x) = C_1 \cos x + C_2 \sin x;$$
$$y = \cos x + C \sin x.$$

All these functions serve as extremal for any *C*. In other words, there is infinite number of solutions.

2.7. Veierstrasse-erdman theorem

Assume, that *y(x)* is solution of Euler equation:

$$F_y - \frac{d}{dx} F_{y'} = 0.$$

If *F* has partial devirative up to second inclusive, then at all points where $\frac{\partial^2 F}{\partial (y')^2} \neq 0$, function *y(x)* has continuous second derivative, and it means, at this point, break is not present. If $\frac{\partial^2 F}{\partial (y')^2} = 0$, then at this point, break is present. Lines composed of extremal piecewises, satisfying condition $\frac{\partial^2 F}{\partial (y')^2} \neq 0$, are named as broken extremal.

Legandr condition. At all points of line *y(x)* supplying extremal to functional *J*, the following condition must be satisfied:

If $F_{y'y'}(x, y, y') \geq 0$ - minimum;

If $F_{y'y'}(x, y, y') \leq 0$ - maximum;

$$x_1 \leq x \leq x_2.$$

***Veierstrasse condition*:** If *y* is minimum (maximum), then:

$$F(x, y, z) - F(x, y, y') - (z - y')F_{y'}(x, y, y') \geq 0, \ (\leq 0),$$

for any z at all points of this range.

2.8. Cased of simplifying or defiation of euler equation

Respects to function $F(x, y, y')$ different cases are possible.

The case №1.

F is independent to *y'*. In this case, Euler equation is represented as:

$$F_y(x, y) = 0.$$

Here, cases often are appeared, when due to combination of boundary conditions equation is unsolved.

<u>Example.</u>

$$J[y(x)] = \int_0^{\pi/2} y(2x - y)dx,\ y(0) = 0,\ y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

Euler equation is:

$$2x - 2y = 0,\ y = x.$$

For this initial conditions, equation has solution, but for other, e.g.,

$$y(0) = 0,\ y(\frac{\pi}{2}) = 1,$$

equation has no solution.

The case №2.

F depends to y' linearly.

$$F(x, y, y') = M(x, y) + N(x, y)y'.$$

Euler equation is turned to more simple:

$$\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x} = 0.$$

After derivation, it is turned from differential to algebraical equation.

Some cases are possible, when in some area this equation is equal to zero identically. It means, that in limits of this area function *J*[*y*] is constant, and variation problem is meaningless.

Example.

$$J[y(x)] = \int_a^b \left(y^2 + 2yy'x\right)dx;$$

$$y(a) = A; \quad y(b) = B;$$

$$\frac{\partial M}{\partial y} = 2y; \quad \frac{\partial N}{\partial x} = 2y; \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \equiv 0.$$

The case №3.

F depends to y only. In this case, Euler equation is represented as:

$$y''Fy'y' = 0.$$

General solution can be taked:

$y = C_1 x + C_2$ - all possible straight lines.

Here C_1, C_2 are arbitrary constants.

Example.

Find extremum of functional (line length between given points).

$$J[y(x)] = \int_a^b \sqrt{1 + y^2(x)}dx;$$

$$y(a) = A; \quad y(b) = B;$$

$$y''(x) = 0; \quad y = C_1 x + C_2; \quad y = \frac{B - A}{b - a}(x - a) + A.$$

(shortest length between two points is straight line).

The case №4.

F is independent to y.

$$F = F(x, y').$$

In this case, Euler equation is turned to:

$\frac{d}{dx}F_{y'}(x, y') = 0; \quad F_{y'}(x, y') = C_1$ - arbitrary constant.

We got common first order differential equation.

Example.

Two points are given $A(1,3)$, $B(2,3)$. Among all possible curves connecting these 2 points, find such points among extremum of the following functional can be reached:

$$J[y(x)] = \int_a^b y'(x)\left[1 + x^2 y'(x)\right]dx.$$

In this case, Euler equation is:

$$\frac{d}{dx}F_y(x, y') = 0; \quad \frac{d}{dx}\left(1 + 2x^2 y'\right) = 0;$$

$$1 + 2x^2 y' = C; \quad y' = \frac{C-1}{2x^2};$$

$$y(x) = \frac{C_1}{x} + C_2; \quad C_1 = \frac{1-C}{2}.$$

Use initial conditions:

$$\begin{cases} 3 = C_1 + C; \\ 5 = \dfrac{C_1}{2} + C_2. \end{cases}$$

hence:

$$y(x) = 7 - \frac{4}{x}.$$

The case №5.

F is independent to x explicitly:

$$F = F(y, y').$$

taking into account, that:

$$\frac{d}{dx}F_y = \frac{dy}{dx}\frac{d}{dy}F_{y'} + \frac{dy'}{dx}\frac{d}{dy'}F_{y'} = y'F_{y'y} + y''F_{y'y'}).$$

In this case, the equation is:

$$F_y - F_{y'y}y' - F_{y'y'}y'' = 0.$$

Multiply by y:

$$y' F_y - y'^2 F_{y'y} - y'' y' F_{y'y'} = 0;$$

$$\frac{d}{dx}\left(F - y' F_{y'}\right) = \frac{dy}{dx} F_y + \frac{d(y')}{dx} F_{y'} - \frac{d(y')}{dx} F_{y'} - y' \frac{dy}{dx} F_{y'y} - y' \frac{d(y')}{dx} F_{y'y'} = 0;$$

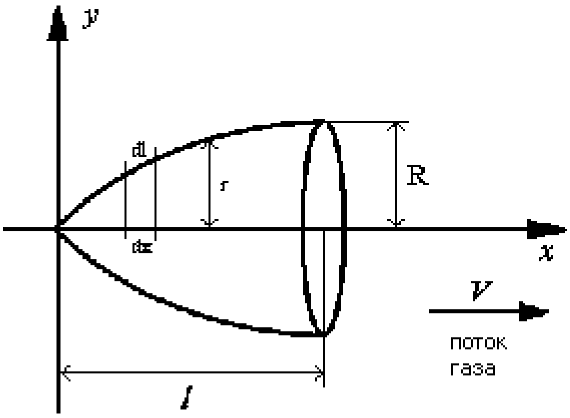
$$\frac{d}{dx}\left(F - y' F_{y'}\right) = 0,$$

where $F - y' F_{y'} = C_1$ - arbitrary constant.

This equation is solved using separation of variables.

Example. Consider a problem:

There is a gas flow, and a solid is moving within it. What shape of the solid is it putted minimal pressure?



If gas density is small enough and we are too far from acoustic speed, then hade is equal to angle of reflection.

$$p = 2\rho V^2 \sin^2 \theta,$$

p – gas density, V – molecules speed relative to the solid, θ – angle of tangent to generatrix and horizontal.

$$dl = (1 + y')^{1/2} dx; \quad r = y(x).$$

A force puts on ring with width dx:

$$dF = 2\rho V^2 \sin^2 \theta \left[2\pi y\left(1 + y'^2\right)^{1/2}\right] \sin\theta dy.$$

Full force is putting along axis OX:

$$F = \int_0^l dF.$$

We shall find a simplified solution, substituting: $\sin\theta = \dfrac{y'}{\left(1 + y'^2\right)^{1/2}} \approx y'$.

Then, a drag is:

$$F = 4\pi\rho V^2 \int_0^l y'^3 \, y dx; \quad y(0) = 0; \quad y(l) = R.$$

Euler equation is:

$$y'^3 - 3\frac{d}{dx}\left(yy'^2\right) = 0.$$

Multiply both sides by *y'*. Left side is become a derivative from expression $y'^3 y$. Integrate the following expression:

$$y'^3 \, y = C; \; y' = \frac{C_1}{\sqrt[3]{y}}; \; y = \left(C_1 x + C_2\right)^{3/4},$$

substituting initial conditions: $y = R\left(\dfrac{x}{l}\right)^{3/4}$. Contour putting minimal pressure to solid is a parabola with the power - 3/4.

2.9. Invariance of euler equation

If functional like:

$$J[y] = \int_a^b F(x, y, y')dx,$$

is turned by substituting independent variable *x* or simultaneously *x* and *y*, then extremal is solving using Euler equation as usual, but it consists of turned equation.

Assume *x* and *y* are function of new variables.

$$x = x(U, V); \quad y = y(U, V).$$

Also, assume that the mutual independence of these functions condition is satisfied.

$$\begin{vmatrix} \dfrac{\partial x}{\partial U}; & \dfrac{\partial x}{\partial V} \\ \dfrac{\partial y}{\partial U}; & \dfrac{\partial y}{\partial V} \end{vmatrix} \neq 0;$$

Then after the substituting:

$$\int F(x,y,y')dx = \int F\left[x(U,V), y(U,V), \frac{\frac{\partial y}{\partial U}+\frac{\partial y}{\partial V}\frac{\partial V}{\partial U}}{\frac{\partial x}{\partial U}+\frac{\partial x}{\partial V}\frac{\partial}{\partial}}\right]\left(\frac{\partial x}{\partial U}+\frac{\partial x}{\partial V}\frac{\partial V}{\partial U}\right)dU =$$

$$= \int \Phi(U,V,V')dU\,.$$

$\Phi(U,V,V')$ - is some new function.

There is a formula to find new extremal.

$$\Phi_V - \frac{d}{dU}\Phi_{V'} = 0\,.$$

Example.

Find extremum of the following functional:

$$J[y] = \int_0^{\ln 2}\left(e^{-x}y'^2 - e^x y^2\right)dx\,.$$

Euler equation for integrand is:

$$y'' - y' + e^{2x}y = 0\,.$$

Do substitute the variables $(x = \ln U;\quad y = V)$. Then, original functional is turned to:

$$J[V] = \int_0^2\left(e^{-\ln U}U^2V'^2 - e^{\ln U}V^2\right)\frac{dU}{U} = \int_0^2\left(V'^2 - V^2\right)dU.$$

For such functional, Euler equation is more simpler:

$$V'' + V = 0;$$

$$V = C_1\cos U + C_2\sin U.$$

Do reverse substituting:

$$y = C_1\cos e^x + C_2\sin e^x\,.$$

Constants are determinated from initial conditions.

2.10. Variation problems in parametric form

In many practical applications, parametric definition of lines is necessary to be used to make calculations simpler.

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}; \qquad t_0 \le t \le t_1 .$$

Assumed, that φ and ψ are continuous and have at least piecewise continuous derivatives. It is necessary both derivatives to be not turned to zero together, that is need the following condition to be satisfied:

$$\varphi'^2 + \psi'^2 = 0 .$$

Each line allow infinite number of parametric representations.

For instance, ellipse can be defined using different kinds of parametric equations:

$$\begin{cases} x = a\cos t; \\ y = b\sin t; \end{cases} \qquad -\pi \le t \le \pi ,$$

$$\begin{cases} x = \dfrac{a\left(1 - z^2\right)}{1 + z^2}; \\ y = \dfrac{2bz}{1 + z^2}. \end{cases} \qquad -\infty \le z \le +\infty .$$

In the case of incorrect way, we can find untrue extremum of functional. In this case, extreamal can be depended from parametric representation instead of *y*. In order to prevent this, it is necessary and enough integrand to do not contain *t* in explicit form. It is necessary the following condition is satisfied.

$$F(x, y, kx', ky') = kF(x, y, x', y'), \text{ k is constant.}$$

If a line L is defined using the following system:

$$\begin{cases} x = \varphi(t); \\ y = \psi(t), \end{cases}$$

where *t* is changing in range $t_0 \le t \le t_1$ and the line L delivers extremum J, then φ and ψ are satisfied the following Euler equations:

$$\begin{cases} \dfrac{dF}{dx} - \dfrac{d}{dt}\left[\dfrac{dF}{dx'}\right] = 0; \\ \dfrac{dF}{dy} - \dfrac{d}{dt}\left[\dfrac{dF}{dy'}\right] = 0. \end{cases}$$

The equations allow to find functions $\varphi(t)$ and $\psi(t)$. Each of those equations is consequence of other equation. For this situation, Veierstrasse form of Euler equation is exist also:

$$\frac{1}{r} = \frac{F_{y'x} - F_{x'y}}{F_1\left(x'^2 + y'^2\right)^{3/2}}; \qquad F_1 = \frac{F_{x'x'}}{y'^2} = \frac{F_{y'y'}}{x'^2} = \frac{F_{y'x'}}{y'x'},$$

where r is radius of curvature of extremal.

Example.

Find extremal of functional:

$$J = \int_{0,0}^{x_1,y_1} y^2 y'^2 \, dx;$$

Turn to parametric form:

$$\begin{cases} x = x(t); \\ y = y(t). \end{cases}$$

Transform integrand in such a way as to exclude depending from t.

$$y^2 y'^2 dx = y^2\left(\frac{dy}{dx}\right)^2 dx = y^2 \frac{\left(\dfrac{dy}{dx}\right)^2}{\left(\dfrac{dx}{dt}\right)^2} \frac{dx}{dt} dt = y^2 \frac{y'^2}{x'^2} x' dt = y^2 \frac{y'^2}{x'} dt.$$

Consider the first Euler equation:

$$F_x = \frac{d}{dx}\left(y^2 \frac{y'^2}{x'}\right) = 0; \quad F_{x'} = \frac{d}{dx'}\left(y^2 \frac{y'^2}{x'}\right) = -\frac{y^2 y'^2}{x'^2};$$

$$\frac{d}{dt}\left(y^2 \frac{y'^2}{x'^2}\right) = 0; \;\; C_2 = 0;; \; y\frac{dy}{dx} = \sqrt{C_1};$$

$$y^2 = 2\sqrt{C_1}x + C_2.$$

It must pass through corresponding boundary points (x_0, y_0)=(0,0). Hence, $C_2 = 0$ and we get:

$$y^2 = \left(\frac{y_1^2}{x_1}\right)x,$$

where y_1, x_1 are point coordinates.

This is parabola equation.

2.11. Summarizing of the simplest problem of calculus of variation

2.11.1. Formulas depend on high order derivatives

Minimization of functional like below:

$$J[y(x)] = \int_{x_0}^{x_1} F\left[x, y, y', \ldots, y^{(n)}\right]dx.$$

Function F must be differentiable with respect to all variables n+2 times. Boundary conditions are set:

$$\begin{cases} y(x_0) = y_0; \\ y'(x_0) = y'_0; \\ y''(x_0) = y''_0; \\ \ldots\ldots\ldots\ldots \\ y^{(n-1)}(x_0) = y_0^{(n-1)}; \end{cases} \qquad \begin{cases} y(x_1) = y_1; \\ y'(x_1) = y'_1; \\ y''(x_1) = y''_1; \\ \ldots\ldots\ldots\ldots \\ y^{(n-1)}(x_1) = y_1^{(n-1)}. \end{cases}.$$

Suppose, boundary conditions are given for both edges. Extremals are defined using Euler- Poisson equation:

$$J[Z(x_1, x_2 \ldots x_N)] = \int_D \ldots \int_D F(x_1, x_2, \ldots x_N, z, p, p_2, \ldots, p_N)dx_1 dx_2 \ldots dx_N.$$

Example.

Find extremal of functional:

$$J[y(x)] = \int_0^1 \left(720x^2 y - y''\right) dx.$$

Boundary conditions are there:

$$y(0) = 0; y'(0) = 1;$$
$$y(1) = 0;\ y'(1) = 1.$$

Euler-Poisson equation is represented as:

$$720x^2 + \frac{d^2}{dx^2}(-2y'') = 0;\ \ y'''' = 360x^2;$$

$$y = x^6 + C_1x^3 + C_2x^2 + C_3x + C_4.$$

Substitute boundary conditions:

$C_1 = -2;\ C_2 = 0;\ \ C_3 = 1;\ \ C_4 = 0$ and with respect to they

$$y(x) = x^6 - 2x^3 + x.$$

2.11.2. Functionals depend on m *functions*

Assume, m functions $y_1(x), y_2(x),\dots, y_m(x)$ are considered.

Boundary conditions must be defined with respect to all functions. Mark them in the following way:

$$y_k(x_0) = y_k^{(0)};\ y_k(x_1) = y_k^{(1)};\ k = 1 \div m.$$

Extremum of functional is necessary to find:

$$J[y_1 ... y_m] = \int_{x0}^{x1} F(x, y_1,..., y_m, y_1',..., y_m')dx.$$

For this, a system of 2nd order differential equations need to be solved.

$$\begin{cases} F_{y1} - \frac{d}{dx} F_{y_1'} = 0; \\ F_{y1} - \frac{d}{dx} F_{y_2'} = 0; \\ ---------- \\ F_{y_m} - \frac{d}{dx} F_{y_m'} = 0. \end{cases}$$

Example.

Find extremum of functional:

$$J[y(x), z(x)] = \int_1^2 (y'^2 + z^2 + z'^2)dx.$$

Boundary conditions are below:

$$y(1) = 1;\ y(2) = 2;\ z(1) = 0;\ z(2) = 1.$$

The system of differential equations for this functional is represented as:

$$\begin{cases} y'' = 0; \\ z - z'' = 0. \end{cases}$$

Solving the system, we get:

$$\begin{cases} y = c_1 x + c_2; \\ z = c_3 e^x - c_4 e^{-x}. \end{cases}$$

For a set c we can get the following expressions:

$$c_1 = 1;$$

$$c_2 = 0;$$

$$c_3 = \frac{1}{e^2 - 1};$$

$$c_4 = \frac{e^2}{e^2 - 1}.$$

Desired extremal is:

$$\begin{cases} y = x; \\ z = \dfrac{sh(x-1)}{sh1}. \end{cases}$$

In general case, boundary conditions may to be not enough to determinate all constants *c*. In this case, some *c* in solution are arbitrary.

2.11.3. Functionals depend on functions of several independent variables

a) In the first, consider functionals depend on functions from 2 variables.

Assume, a function $Z(x, y)$ depends on 2 variables. Physically meaning of $Z(x, y)$ is some arbitrary surface. Such way, corresponding functional can be written as:

$$J[Z(x,y)] = \iint_D F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dxdy.$$

Problem has solution, if function *F* is able to be derivatived three time with respect to all its arguments. Suppose desired function *Z* in area *D* is continuous together with its derivatives upto 2[nd] order (inclusive). Assume, area *D* has edge *Γ*. Here, we are forced to define boundary conditions at all area *Γ*. Surface $Z(x, y)$ provides extremum of functional, if it is satisfied Euler-Ostrogradski equation:

$$F_z - \frac{\partial}{\partial x}\{F_p\} - \frac{\partial}{\partial y}\{F_q\} = 0,$$

where: $p = \dfrac{\partial z}{\partial x}$; $q = \dfrac{\partial z}{\partial y}$.

$$\frac{\partial}{\partial x}\{F_p\} = F_{px} + F_{pz}\frac{\partial z}{\partial x} + F_{pp}\frac{\partial p}{\partial x} + F_{pq}\frac{\partial q}{\partial x};$$

$$\frac{\partial}{\partial x}\{F_q\} = F_{qy} + F_{qz}\frac{\partial z}{\partial y} + F_{qp}\frac{\partial p}{\partial y} + F_{qq}\frac{\partial q}{\partial y}.$$

This equation is used to solve extremals.

Example.

Find extremum of functional like:

$$J[Z(x,y)] = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 \right] dxdy .$$

Integrand is:

$$F(x,y,z,p,q) = p^2 - q^2 .$$

Hence, we easy find Euler-Ostrogradski equation:

$$-\frac{\partial}{\partial x}(2p) - \frac{\partial}{\partial y}(-2q) = 0 ;$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0 .$$

Further, we find solution in common way.

b) Assume, desired function *Z* is function depending on *N* variables:

$$Z = Z(x_1, x_2, ..., x_N).$$

We have functional:

$$J[Z(x_1, x_2 ... x_N)] = \int_D ... \int_D F(x_1, x_2, ... x_N, z, p, p_2, ..., p_N) dx_1 dx_2 ... dx_N ;$$

$$p_k = \frac{\partial z}{\partial x_k} ; \; k = 1 \div n .$$

Euler-Ostrogradski equation is:

$$F_z - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ F_{p_i} \right\} = 0 .$$

In detailed representation:

$$F_z - \sum_{i=1}^{n} (F_{x_i p_i} + F_{z p_i} p_i + F_{p_i p_i} \frac{\partial p_i}{\partial x_i}) = 0 .$$

In this case, *Γ* isn't line, but it is some multidimensional bound of multidimensional area.

2.12. Variation problems with conditional extremum

Variation problem, which is solving extremum of functional with additional conditions to desired function, is named as variation problem with conditional extremum.

2.12.1.Isoperimetric problem

Two functions are given: $F(x, y, y'), G(x, y, y')$. Supposing that they have continuous partial derivatives for 1st and 2nd order in considered range $x_0 \le x \le x_1$, for any y and y'. Assume, functional $K[y]$ is defined using the following expression:

$$K[y] = \int_{x0}^{x1} G(x, y, y')dx = l, \qquad (2.12.1)$$

where l is given value.

For these conditions, it is necessary to determinate extremum of functional J.

$$J[y] = \int_{x0}^{x1} F(x, y, y')dx \to extr. \qquad (2.12.2)$$

To solve this problem, **Euler theorem** is used:

If a curve $y = y(x)$ provides conditional extremum to functional:

$$J[y] = \int_{x0}^{x1} F(x, y, y')dx,$$

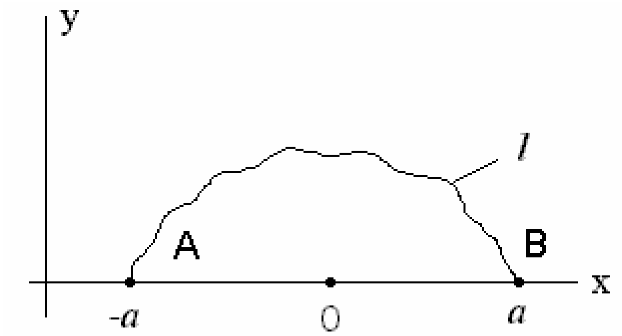
with condition

$$K[y] = \int_{x0}^{x1} G(x, y, y')dx = l,\ y(x_0) = y_0,\ y(x_1) = y_1,$$

and $y(x)$ is not extremal of functional $K[y]$, then such constant λ exists, that a curve $y(x)$ is conditionless extremal of new functional L:

$$L = \int_{x0}^{x1} [F(x, y, y')dx + \lambda G(x, y, y')]dx.$$

<u>Example.</u>



Line connects points A and B.

$$y(-a) = y(a) = 0.$$

Length of line is given: πa< *l* <2a.

It is required to find function $y(x)$ in order to maximal area embraced by curve *l*.

Solution.

The problem comes to finding extremum of expression:

$$J[y(x)] = \int_a^b y(x)dx \text{ with condtions;}$$

$$y(-a) = y(0) = 0,$$

and with additional condition

$$K[y] = \int_{-a}^{a} \sqrt{1 + y'^2}dx = l.$$

Compose auxilary function:

$$H = F + \lambda G = y + \lambda\sqrt{1 + y'^2}.$$

We get new functional:

$$L = \int_{-a}^{a} H(x, y, y')dx.$$

Euler equation is:

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right)=1.$$

Further:

$$\frac{\lambda y'}{\sqrt{1+y'^2}}=x+c_1.$$

After transformation, we get:

$(x+c_1)^2+(y+c_2)^2=\lambda^2$ - is equation of piece of circle.

2.12.2. The rule of mutuality of isoperimetric problems

Extremal y_1 satisfying conditions:

$$\begin{cases} J[y]\to extr; \\ K[y]=const, \end{cases}$$

is congruent with extremal y_1 satisfying conditions:

$$\begin{cases} K[y_1]\to extr; \\ J[y_1]=const. \end{cases}$$

2.12.3. Isoperimetric problems with several conditions

If piecewise smooth function $y(x)$ provides conditional extremum of functional $J_0[y]$ with the following conditions:

$$\begin{cases} J_1[y]=l_1; \\ J_2[y]=l_2; \\ ------ \\ J_k[y]=l_k; \end{cases}$$ $l_1 \div l_k$ – are assigned values.

$$(J_i[y]=\int_{x0}^{x1}F_i(x,y,y')dx).$$

then a set of constants $\{\lambda_i\}$, $i = 0 \div k$, $\lambda_0^2 + \lambda_1^2 + ... + \lambda_k^2 = 1$, exists in such a way, that a curve *y* provides conditionless extremum of functional:

$$L = \int_{x0}^{x1} (\lambda_0 F_0 + \lambda_1 F_1 + \lambda_2 F_2 + ... + \lambda_k F_k) dx .$$

2.12.4. Isoperimetric problems for a set of functions

Isoperimetric problem is named in that case, if it is required to find extremum of functional:

$$J[y] = \int_{x0}^{x1} F(x, y_1, y_2 ... y_n, y_1', y_2' ... y_n') dx ,$$

with conditions:

$$\begin{cases} \int_{x0}^{x1} G_1(x, y_1 ... y_n, y_1' ... y_n') dx = l_1; \\ \int_{x0}^{x1} G_2(x, y_1 ... y_n, y_1' ... y_n') dx = l_2; \\ ------------------ \\ \int_{x0}^{x1} G_m(x, y_1 ... y_n, y_1' ... y_n') dx = l_m . \end{cases}$$

$l_l \div l_m$ – are assigned values.
(Continuous requirements are the same)

To find solution a functional is composed:

$$\Phi[y_1 \div y_n] = \int_{x0}^{x1} \left(F_0 + \sum_{i=1}^{m} \lambda_i G_i \right) dx .$$

It is solved as usually (finding conditionless extremum).

Constants λ and C is determinated using boundary and isoperimetric conditions.

Example.

Find extremal of functional like that:

$$J[y(x),Z(x)]=\int_0^1(y'^2+z'^2-4xz'-4z)dx;$$

$$\begin{cases} y(0)=0, \quad z(0)=0; \\ y(1)=1, \quad z(1)=1, \end{cases}$$

with additional condition:

$$\int_0^1(y'^2-xy'-z'^2)dx=2.$$

Compose auxilary functional:

$$\Phi=\int_0^1[y'^2+z'^2-4xz'-4z+\lambda(y'^2-xy'-z'^2)]dx.$$

Corresponding Euler equations are there:

$$\begin{cases} \dfrac{d}{dx}(2y'+2\lambda y'-\lambda x)=0; \\ -4-\dfrac{d}{dx}(2z'-4x-2\lambda z')=0. \end{cases}$$

Solution is:

$$\begin{cases} y(x)=\dfrac{\lambda x^2+2c_1x}{4(1+\lambda)}+c_2; \\ z(x)=\dfrac{c_3x}{2(1-\lambda)}+c_4. \end{cases}$$

Taking into account boundary conditions, we get the following:

$$c_1=\frac{3\lambda+4}{2},\ c_2=0,\ c_3=2(1-\lambda),\ c_4=0.$$

After the substituting:

$$\begin{cases} y(x)=\dfrac{\lambda x^2+(3\lambda+4)x}{4(1+\lambda)}; \\ z(x)=x. \end{cases}$$

After repeated using isoperimetric condition:

$$\frac{1}{3}(23\lambda^2+46\lambda+24)=48(\lambda^2+2\lambda+1).$$

Hence:

$$\lambda_1=-\frac{10}{11}.$$

(Other root is $\lambda_2=-\frac{12}{11}$ - is not satisfied original isoperimetric condition).

Finally:

$$\begin{cases} y(x)=\dfrac{7x-5x^2}{2}; \\ z(x)=x. \end{cases}$$

2.12.5. Lagrange problem

(It is problem for conditional extemum also)

Target setting.

Find functions $y_1, y_2, ..., y_n$ providing extremum of functional, with boundary conditions:

$$y_j(x_0)=y_{j0};\ \ y_j(x_1)=y_{j1};\ \ j=1\div n.$$

Additional conditions do not concern to functional of desired functions, but they concern to relations between these original functions, and are represented as:

$$\begin{cases} \varphi_1=\varphi_1(x,y_1,y_2,...,y_n)=0; \\ \varphi_2=\varphi_2(x,y_1,y_2,...,y_n)=0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi_m=\varphi_m(x,y_1,y_2,...,y_n)=0. \end{cases} \qquad m<n.$$

To find solution the following theorem is used.

***Theorem**:* Functions $y_1, y_2, ..., y_n$, implemting extremum of functional J with a set of conditions φ_i, $i=1 \div m$, are satisfied Euler equations for modifed the following functional, if multipliers $\lambda_i(x)$, $i=1 \div m$ are choosen accordingly:

$$J^* = \int_{x_0}^{x_1} \left[F + \sum_{i=1}^{m} \lambda_i(x) \varphi_i \right] dx .$$

(Here λ_i – are not constant already, but they are functions from x)

$\varphi_i = 0$ can be considered as Euler equations for functional J^*, if arguments of the functional are considered not only functions $y_1(x) \div y_n(x)$, but and additional functions $\lambda_1(x) \div \lambda_m(x)$ too. Mark:

$$F + \sum_{i=1}^{m} \lambda_i \varphi_i = \Phi(x, y_1, ..., y_n, y'_1, ..., y'_n).$$

Then, functions $y_j(x)$, and functions $\lambda_i(x)$ are determinated from joint solution of the following system of equations:

$$\begin{cases} \Phi_{y_i} - \dfrac{d}{dx} \Phi_{y'_j} = 0; & (j = 1...n); \\ \varphi_i(x, y_1, ..., y_n) = 0; & (i = 1...m). \end{cases}$$

($n+m$ equations for $n+m$ desired unknown functions)

Example.

The surface, satisfying the following equation, is given:

$$15x - 7y + z - 22 = 0 .$$

At that two points: $A(1;-1;0)$; $B(2;1;-1)$ are given. Find equation of curve with minimal length connecting these points.

Solution.

At any surface, satisfying equation $\varphi(x, y, z) = 0$, length between points $A(x_0, y_0, z_0)$; $B(x_1, y_1, z_1)$ are determinated using formula:

$$l = \int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx ;$$ $y=y(x)$; $z=z(x)$ – are projection of line connecting these points at corresponding coordinate planes.

Compose auxilary functional like that:

$$J^{*} = \int_{1}^{2} [\sqrt{1 + y'^2 + z'^2} + \lambda(x)(15x - 7y + z - 22)]dx.$$

According Euler equations are below:

$$\begin{cases} \lambda(x)(-7) - \dfrac{d}{dx}\left(\dfrac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0; \\ \lambda(x) \cdot 1 - \dfrac{d}{dx}\left(\dfrac{z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0. \end{cases}$$

After transformation:

$$\frac{d}{dx}\left(\frac{y' - 7z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0;$$

$$\frac{y' - 7z'}{\sqrt{1 + y'^2 + z'^2}} = C_1; \;\; z' = 7y' - 15;$$

$$y(x) = C_3 x + C_2; \;\; C_3 = 2; \;\; C_2 = -3;$$

$$\begin{cases} y(x) = 2x - 3; \\ z(x) = 1 - x. \end{cases}$$

We get equation of right line, connecting two points at the surface (the surface is defined by isoperimetric condition).

$$\lambda(x) \equiv 0 \;.$$

Length l is equal: $l = \sqrt{6}$.

Shortest line at given surface and connecting two specified points is named as a geodesic line.

2.13. Variation tasks with moving boundary

This is class of the tasks when Limits of integral in function are not constants.

2.13.1. The simple task with moving boundary

Let $F(x, y, y')$ - Thrice differentiable function on all arguments and in plane XOY two curves are given:

$$y = \varphi(x);$$
$$y = \psi(x).$$

Let's consider functional:

$$J[y] = \int_{\gamma} F(x, y, y')dx.$$

Given functional determined will be considered in a class of curve *y(x)* What is the ends lay on these lines *φ(x)* and *ψ(x)*. Thus $y_0=\varphi(x_0)$, $y_1=\psi(x_1)$,but x_0 and x_1. unknown. It is required to find extremum initial functional
For solution we shall use the following theorem, Let the curve *y (x)* gives extremum of function:

$$J[y] = \int_{\gamma} F(x, y, y')dx.$$

Among all curves, two given lines *φ (x) and ψ (x)* connecting two any point. Then *y (x)* is extremum and on its ends *A (x0, y0, z0); B*(x_1, y_1, z_1).

Conditions transversally a kind are satisfied:

$$\begin{cases} \left[F + (\varphi' - y')F_{y'}\right]_{x=x_0} = 0; \\ \left[F + (\psi' - y')F_{y'}\right]_{x=x_1} = 0. \end{cases}$$

These conditions were used for find extremum. The solution with use of the theorem is carried out by the following sequence of actions:

1. To write and solve appropriate equation Eiler a usual way. Consider moving boundary, thus we find $y = f(x, c_1, c_2)$, c_1, c_2 - const.

2. Using two equations transversal and two new equations

$$f(x_0, c_1, c_2) = \varphi(x_0);$$

$$f(x_1, c_1, c_2) = \psi(x_1).$$

We find the system from four equations with four unknown const:

$$c_1, c_2, x_0, x_1.$$

3. Solution this system we are finding the const c_1, c_2, x_0, x_1.

<u>Example:</u>

Find the shortest distance between two lines which are given by the following equations:

$$y = x^2,\ x - y = 5.$$

Solution.

We are finding the value of extremum function:

$$J = \int_{x0}^{x1} \sqrt{1 + y'^2}\, dx;$$

$$\varphi(x) = x^2;$$

$$\psi(x) = x - 5.$$

Solve initial equation Eiler, Including boundary points as though fixed:

$$y = c_1 x + c_2.$$

The condition transversally for this situation has the following kind:

$$\begin{cases} \left[\sqrt{1+y'^2} + (2x - y')\dfrac{y'}{\sqrt{1+y'^2}}\right] = 0, & \text{при } x = x_0; \\ \left[\sqrt{1+y'^2} + (1 - y')\dfrac{y'}{\sqrt{1+y'^2}}\right] = 0, & \text{при } x = x_1; \end{cases}$$

$$\begin{cases} c_1 x_0 + c_1 = x_0^2; \\ c_1 x_1 + c_1 = x_1 - 5; \end{cases}$$

$$y' = c_1;$$

$$\begin{cases} \sqrt{1+c_1^2} + (2x_0 - c_1)\dfrac{c_1}{\sqrt{1+c_1^2}} = 0; \\ \sqrt{1+c_1^2} + (1 - c_1)\dfrac{c_1}{\sqrt{1+c_1^2}} = 0; \end{cases}$$

$$c_1 = -1; \quad c_2 = \frac{3}{4};$$

$$x_0 = \frac{1}{2}; \quad x_1 = \frac{23}{8}.$$

Extremum it is achieved on function $y = -x + 3/4$. Thus the minimal distance is equal:

$$l = \int_{1/2}^{23/8} \sqrt{1+(-1)^2}\,dx = \frac{19\sqrt{2}}{8}.$$

2.13.2. A task for three measured spaces

For this task line located in measured spaces, i.e we need to find functional kind:

$$J[y,z] = \int_{x0}^{x1} F(x,y,z,y',z')dx.$$

Let even one of boundary points (*x0*, *y0*, *z0*) or *B* (*x1*, *y1*, *z1*) moves on the given curve.

Then extremum of function may be achieved only on curves satisfying system of equations Eiler.

$$\begin{cases} F_y - \dfrac{d}{dx}F_{y'} = 0; \\ F_z - \dfrac{d}{dx}F_{z'} = 0. \end{cases}$$

For simplicity we shall consider, that point A is fixed motionlessly, and the point B may move over a curve which is set by system of the equations.

$$\begin{cases} y = \varphi(x); \\ z = \psi(x); \end{cases}$$

$$A(x_0, y_0, z_0),\ B(x, y, z).$$

In this case the condition transversal will become:

$$F + (\varphi' - y')F_{y'} + (\psi' - Z')F_{z'} = 0, \quad \text{when } x = x_1.$$

If also the point A moves over a curve it means, that position of a point A can be determined system:

$$\begin{cases} y = \tilde{\varphi}(x); \\ z = \tilde{\psi}(x). \end{cases}$$

The condition transversally for a point A looks like:

$$F - (\tilde{\varphi} - y')F_{y'} + (\tilde{\psi} - Z')F_{z'} = 0, \quad \text{when} x = x_0.$$

Example.

Find the shortest distance from a point $M(x_0, y_0, z_0)$ up to the straight line any way focused in space given by system of the equations:

$$\begin{cases} y = mx + p; \\ z = nx + q. \end{cases}$$

Solution

The problem is reduced to search of integral:

$$J[y, z] = \int_{x0}^{x1} \sqrt{1 + y'^2 + z'^2}\, dx.$$

When the condition that x_1 Moves on a line described by system:

$$\begin{cases}\varphi(x) = mx + p; \\ \psi(x) = nx + q.\end{cases} \tag{2.13.1}$$

The common solution for (13.1) in this case looks like:

$$\begin{cases}y = c_1x + c_2; \\ z = c_3x + c_4.\end{cases}$$

For the right border of a condition transversally:

$$\sqrt{1+y'^2+z'^2} + (m-y')\frac{y'}{\sqrt{1+y'^2+z'^2}} + (n-z')\frac{z'}{\sqrt{1+y'^2+z'^2}} = 0, \quad \text{when} x = x_1.$$

Let's take into account, that: $y' = c_1, z' = c_3$.

Substituting it in a condition transversal, we receive $1 + mc_1 + nc_3 = 0$.

It is necessary take into account that unknown extremum should pass through point $M(x_0, y_0, z_0)$.

So we find new system of equations:

$$\begin{cases}y_0 = c_1x_0 + c_2; \\ z_0 = c_3x_0 + c_4.\end{cases}$$

Other end moves over a straight line, the point means x_1. It is connected by system:

$$\begin{cases}c_1x_1 + c_2 = mx_1 + p; \\ c_3x_1 + c_4 = nx_1 + q.\end{cases}$$

Thus, there are 5 equations and 5 unknown x_1, c_1, c_2, c_3, c_4. Solving these equations, we receive:

$$x_1 = \frac{x_0 + m(y_0 - p) + n(z_0 - q)}{1 + n^2 + m^2};$$

$$c_1 = \frac{mx_0 + mn(z_0 - q) - (1 + n^2)(y_0 - p)}{m(y_0 - p) + n(z_0 - q) - (m^2 + n^2)x_0};$$

$$c_3 = \frac{nx_0 + mn(y_0 - p) - (1+m^2)(z_0 - q)}{m(y_0 - p) + n(z_0 - q) - (m^2 + n^2)x_0}.$$

C2 and *C4* not necessarily to find.

Answer:

$$\min J[y,z] = \sqrt{x_0^2 + (y_0 - p)^2 - \frac{x_0 + m(y_0 - p) + n(z_0 - q)}{1 + n^2 + m^2}}.$$

Let one of points is fixed - $A(x_0, y_0, z_0)$, other point may move on some surface which equation is set by the equation, In this case the condition transversal becomes:

$$\begin{cases} \left[F - y'F_{y'} + (\varphi'_x - z')F_{z'}\right]_{x=x_1} = 0; \\ \left[F_{y'} + F_{z'}\varphi'_y\right]_{x=x_1} = 0. \end{cases}$$

This conditions together with the equation $z = \varphi(x, y)$ enable to find two arbitrary constant in equation Elier, other two constant can determine from conditions extremum by fixed point A.

Example.

The point *A* (1,1,1) is given, the sphere which surface is described by the equation is given $x^2 + y^2 + z^2 = 1$. To find the shortest distance from a point up to sphere.

Solution.

The task is reduced to research on extremum following functional:

$$J[y,z] = \int_{x_1}^{1} \sqrt{1 + y'^2 + z'^2}\,dx.$$

Extremum in a general view it is given by the following system of the equations:

$$\begin{cases} y = C_1 x + C_2; \\ z = C_3 x + C_4. \end{cases}$$

From a condition of passage extremum through a point A (1,1,1), we shall receive:

$$\begin{cases} C_1 + C_2 = 1; \\ C_3 + C_4 = 1. \end{cases}$$

The condition transversally will become:

$$\begin{cases} \left[\sqrt{1+y'^2+z'^2} - \dfrac{y'^2}{\sqrt{1+y'^2+z'^2}} + \left(\dfrac{-x}{\sqrt{1-x^2-y^2}} - z' \right) \cdot \dfrac{z'}{\sqrt{1+y'^2+z'^2}} \right] \Bigg|_{x=x_1} = 0; \\ \left[\dfrac{y'}{\sqrt{1+y'^2+z'^2}} + \dfrac{z'}{\sqrt{1+y'^2+z'^2}} \cdot \dfrac{(-y)}{\sqrt{1-x^2-y^2}} \right] \Bigg|_{x=x_1} = 0. \end{cases}$$

From the condition transversally we find the following:

$$\begin{cases} z_1 - C_3 x_1 = 0; \\ C_1 z_1 - C_3 y_1 = 0. \end{cases}$$

where: x_1,y_2,z_1- coordinate of point B.

$$\begin{cases} y_1 = C_1 x_1 + C_2; \\ z_1 = C_3 x_1 + C_4. \end{cases}$$

$C_1 = 1;\ C_2 = 0; C_3 = 1; C_4 = 0.$

It follows that:

$$\begin{cases} y(x) = y = x; \\ z(x) = z = x. \end{cases}$$

Having substituted it in the equation of sphere, we shall find:

$$x^2 + y^2 + z^2 = 1;$$

$$x_1^2 + y_1^2 + z_1^2 = 1;$$

$$x_1 = y_1 = z_1 = \pm \frac{1}{\sqrt{3}}.$$

$$B_1\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right);$$

$$B_2\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right).$$

Answer: $$B_1\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right);$$

$$J_{\min} = \int_{\frac{1}{\sqrt{3}}}^{1}\sqrt{1+1+1}dx = \int_{\frac{1}{\sqrt{3}}}^{1}\sqrt{3}dx = \sqrt{3}-1.$$

2.14. Geodesic distance

Geodesic distance: Length of a geodetic line between two given points(J-length).

Furthermore, a geodetic line in curvilinear space sometimes name J - a straight line.

Example.

The points A (0, 0) and B (1, 1) were given to be on some curvilinear surface, the distance on a surface is determined by expression:

$$J[y] = \int_{A}^{B} y^2 \cdot y'dx.$$

It is required to find geodetic length between these points on a plane.

Solution.

Geodetic distance - the minimal distance, i.e. min *J [y]*, boundary conditions: coordinates A and B.

We work out equation Эйлера:

$$2 \cdot y^2 \cdot y' - \frac{d}{dx}(2 \cdot y^2 \cdot y') = 0;$$

$$yy'' + y'^2 = 0 = \frac{d}{dx}(yy').$$

From this equation we can write down $yy' = C = const$.

After some transformations:

$$y^2 = C_1 \cdot x + C_2 .$$

Using boundary conditions:

$$\begin{cases} y(x=0) = 0; \\ y(x=1) = 1; \end{cases}$$

$$C_1 = 1; C_2 = 0.$$

Thus ,the equation extremum:

$$y^2 = x; y = \sqrt{x} .$$

From this it follows that:

$J(A,B) = 0,25;$

geodesic length= 0,25.

The geodetic distance between a point and a line is determined more difficultly. Here it is necessary to observe simultaneously two conditions:

1) $\min \int_A^B F(x, y, y')dx$.
2) That point in which our geodetic line and the given line are mutually perpendicular gets out.

Geodetic distance between a point and a line - L - distance lengthways extremum, connecting a point and line L in that place where extremum and line L are crossed perpendicularly.

Geodetic circle (J - a circle) - the line, which all points are on identical distance from the given point.

2.15. Explosive problems

Earlier determined function $F(x, y, y')$. As twice differentiable on *x*.

Parameter of it was a condition $F_{y'y'} \neq 0$.

However there is a class of problems which at such severe constraints have no the decision, but at mitigation of conditions well are solved.

Thus such methods allow to find extremum functional as piecewise continuous function.

2.15.1. Explosive problems of the first sort

Let's consider some functional:

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')dx.$$

Boundary condition: Let all allowable decisions satisfy to conditions:

$$\begin{cases} y(x_0) = y_0; \\ y(x_1) = y_1. \end{cases}$$

But except for it we admit, that the required decision *y (x) may* have a break in some point $x_0 < C < x_1$. This break may be only there.

Where it is carried out:

$$F_{y'y'} = 0.$$

For search of the decision we shall take advantage of condition Вейерштрасса - Эрдмана:

$$\begin{cases} F_{y'}\big|_{x=C-0} = F_{y'}\big|_{x=C+0} = 0; \\ (F - y'F_{y'})\big|_{X=C-0} - (F - y'F_{y'})\big|_{X=C+0} = 0. \end{cases}$$

If C - a point of a break from the different sides from a point C function y may be expressed by various formulas.

$C - 0$ and $C + 0$ - pieces of function y on the different sides from a break.

Except for it, itself extremum it should be continuous.

$$y(x \to C - 0) = y(x \to C + 0).$$

Set of these conditions allows to find extremum and coordinates of a point of a break C.

Example.

It is given functional:

$$J[y] = \int_0^2 (y'^2 - y^2)dx.$$

Find extremal:

$F_{y'y'} = 2 > 0$; $F_{y'} = 2y'$ - means, in this example the solution can be found and as smooth function.

Example.

It is given functional:

$$J[y] = \int_0^2 (y'^4 - 6y'^2)dx.$$

Boundary conditions:

$$y(0) = 0;\ y(2) = 0.$$

$F_{y'y'} = 12y' - 12$ - points quite may find.

$$F_{y'y'} = 0.$$

Means, at extremum presence of breaks in any point is possible.

Let's search extremum as a broken line.

Sub integral function depends only from y'. From here the solution will be direct lines $y = C1x + C2$.

In the field of smooth functions the decision one: $y = 0$.

$$y(0) = 0 = C_1 \cdot 0 + C_2 \Rightarrow C_2 = 0;$$

$$y(2) = 0 = C_1 \cdot 2 + C_2 \Rightarrow (C_2 = 0) = C_1 = 0.$$

Therefore in the given task the not trivial decision is possible only among extremum with a break.

F - it is identical in both parts, that y_+ and y_-. There will be direct lines, but with the factors:

$$\begin{cases} y_+ = px + q; (x_0 \le x \le C); \\ y_- = mx + n; (C \le x \le x_1). \end{cases}$$

m, n, p, q, C – unknown.

If we substitute boundary condition then n = 0; q = -2p.

$$\begin{cases} y_- = mx; \\ y_+ = p(x+2). \end{cases}$$

At the same time extremum it should be continuous in point C:

$$y_- = y_+;$$
$$y_-(x = C) = y_+(x = C) \Rightarrow mC = p(C-2).$$

Using condition Вейерштрасса – Эрдмана:

$$\begin{cases} F_{y'} = 4y'^3 - 12y'; \\ F - y'F_{y'} = -3y' + 6y'^2. \end{cases}$$

It is possible to notice $y'_- = m; y'_+ = p$.

Let's substitute it in condition В-Э. We shall receive:

$$\begin{cases} 4m^3 - 12n = 4p^3 + 12p; \\ -3m + 6n^2 = -3p^4 + 6p^2. \end{cases}$$

After transformation:

$$\begin{cases} (m-p)\cdot(m^2 + mp + p^2 - 3) = 0; \\ (m^2 - p^2)\cdot(m^2 + p^2 - 2) = 0. \end{cases}$$

From the second equation $m = p; m = -p; m^2 + p^2 = 2$. If $m = p$, extremum has a continuous derivative and earlier this variant is rejected. It is means, both equations can be divided on $m - p$.

$$\begin{cases} m^2 + m\cdot p + p^2 + 3 = 0; \\ (m+p)\cdot(m^2 + p^2) = 2. \end{cases}$$

This system has more simple solution:

$$1: \begin{cases} m + p = 0; \\ m^2 + mp + p^2 = 3; \end{cases}$$

$$2: \begin{cases} m^2 + p^2 = 2; \\ m^2 + mp + p^2 = 3. \end{cases}$$

The equations of system 2 are received when $m = -p$. Also are neglected.

The system 1 has two solutions:

$$1: m = \sqrt{3}; p = -\sqrt{3};$$
$$2: m = -\sqrt{3}; p = \sqrt{3}.$$

Taking into consideration, that $m = -p$. And substituting in conditions $y_-(C) = y_+(C)$. We receive $C = 1$.

There are two sought extremum. Completely equal in rights:

$$y_I = \begin{cases} \sqrt{3}x; & 0 \le x \le 1; \\ -\sqrt{3}(x-2); & 1 \le x \le 2; \end{cases}$$

$$y_{II} = \begin{cases} -\sqrt{3}x; & 0 \le x \le 1; \\ \sqrt{3}(x-2); & 1 \le x \le 2. \end{cases}$$

2.15.2. Explosive problems of the second sort

$$J = \int_{x_1}^{x_2} F(x, y, y')dx; \qquad y(x_1) = y_1; \qquad y(x_2) = y_2.$$

Problems of the second sort - when function F has break.

Let this break lengthways $y = \Phi(c)$ - a curve. Let on the one hand from result $F_1(x, y, y')$, with another $F_2(x, y, y')$. If the solution exists, it too consists of pieces extremum. Thus both extremums have the general point on a line of break.

$$x_1 < c < x_2; \quad x = c; \; y = \Phi(c).$$

For determination of a sought broken line extremum we receive two equations Elier. They contain four constants c_1, c_2, c_3, c_4. Also it is necessary to find unknown C where meet 2 extremums.

Two boundary condition:

$$y(x_1) = y_1, y(x_2) = y_2;$$

$$y_1(x = c) = \Phi(c);$$

$$y_2(x = c) = \Phi(c).$$

Condition on a joint:

$$F_1 + (\Phi' - y')F_{1y'}\Big|_{x=c-0} = F_2 + (\Phi' - y')F_{2y'}\Big|_{x=c+0}.$$

2.15.3. Explosive problems for extremum *from several functions*

$$F(x, y_1, y_2, ... y_n, y_1', y_2', ... y_n').$$

If sub integral function *F* is continuous on all arguments and has private derivatives up to the third order for realization of broken lines extremum *y* a condition should be satisfied Вейерштрасса – Эрдмана.

$$\left.\frac{\partial F}{\partial y_i'}\right|_{x_0-0} = \left.\frac{\partial F}{\partial y_i'}\right|_{x_o+0};$$

$$F - \sum_{i=1}^{n} y_i' \left.\frac{\partial F}{\partial y_i'}\right|_{x_0-0} = F - \sum_{i=1}^{n} y_i' \left.\frac{\partial F}{\partial y_i'}\right|_{x_0+0}.$$

2.16. One-sided variations

$$J[y] = \int_{x_1}^{x_2} F(x, y, y')dx; \qquad y(x_1) = y_1; \qquad y(x_2) = y_2.$$

To find extremum it functional under certain conditions. If earlier the condition was set by the equation, now an inequality:

$$y - \phi(x) \geq 0.$$

At such formulation required extremumwill consist of pieces of borders and $\phi(x)$ and from pieces extremum y. In points of a joint $\phi(x)$ and y the basic extremum and may have explosive points.

$$\left\lfloor F(x, y, y') - F(x, y, \phi') - (\phi' - y')F_{y'}(x, y, y') \right\rfloor_{x=x_c} = 0,$$

x_c - factor of a point of a joint.

If in x_c $F_{y'y'} \neq 0$, that extremum concerns border ϕ.

Example.

A (-2, 3), *B* (2, 3).

Find the shortest way between *A* and *B* which lays below a parabola $y \le x^2$.

$$J = \int_{-2}^{2} \sqrt{1 + y'^2} dx = \min;$$

$$y \le x^2;$$

$$y(-2) = 3;$$

$$y(2) = 3;$$

$$y = C_1 + C_2 x;$$

$$F_{y'y'} = \frac{1}{[1 + y'^2]^{3/2}} \ne 0.$$

In a point of a contact of ordinate of a parabola and ordinates of straight lines coincide:

$$\begin{cases} C_1 + C_2 x_c = x_c^2; \\ C_2 = 2x_c; \end{cases}$$

$$C_1 + 2C_2 = 3;$$

$$y = \begin{cases} -2x - 1, & -2 \le x \le -1; \\ x^2, & -1 \le x \le 1; \\ 2x - 1, & 1 \le x \le 2. \end{cases}$$

In a case of more complex borders the mobile ends are used.

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ИСЧИСЛЕНИЕ В РАДИОЭЛЕКТРОНИКЕ

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на английском языке

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Ответственный за выпуск – зав. кафедрой профессор О. Р. Никитин

Печатается в авторской редакции
Технический редактор Н. В. Тупицына
Компьютерная верстка А. С. Марютина, С. С. Кормина,
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Подписано в печать 10.10.06
Формат 60х84/16. Бумага для множит. техники. Гарнитура Таймс.
Печать на ризографе. Усл. печ. л. 5,81. Уч.-изд. л. 4,99. Тираж 100 экз.
Заказ
Издательство
Владимирского государственного университета
600000, Владимир, ул. Горького, 87